Лаб. 7. Control Lyapunov function (Функція Ляпунова керування).

Consider systems of the form $\dot{x} = f(x, u)$. We will not investigate in-depth topics such as Input-to-State-Stability (ISS) and Input-Output Stability (IOS). Instead we will study how control is used to obtain desired stability as pertitent to applications in robotics.

At a basic level, our goal is to obtain u in a feedback-form, i.e.

$$u = \phi(x),$$

so that the resulting *closed-loop* systems has the dynamics

$$\dot{x} = f(x, \phi(x))$$

Example 7. *1-d examples.* Consider the system

$$\dot{x} = ax^2 - x^3 + u, \text{ for some } a \neq 0$$

The simplest approach is to set

$$u = -ax^2 + x^3 - x$$

which results in the closed-loop system

$$\dot{x} = -x$$

which is exponentially stable. This approach was to simply cancel all nonlinear terms. But actually, it is not really necessary to cancel the term $-x^3$ since it is already dissipative. A more economical control law would have just been:

$$u = -ax^2 - x$$

The question of determining a proper u also comes down to finding a Lyapunov function. One approach is to actually specify the Lyapunov function V and a negative definite $\dot{V}$ and then find u to match these choices. For instance, in the example above, let

$$V(x) = \frac{1}{2}x^2$$

and let

$$\dot{V} = ax^3 - x^4 + xu \leq -L(x),$$

for some positive definite $L(x)$. One choice is $L(x) = x^2$ which results in

$$u = -ax^2 + x^3 - x,$$

i.e. the same expensive control law. But another choice is to include higher-order terms, i.e. $L(x) = x^2 + x^4$. Then we have

$$u = -ax^2 - x,$$

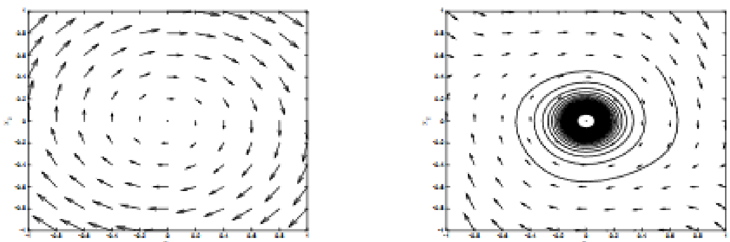
Example: $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_1 u \quad$ and $\quad V(x) = \frac{1}{2}(x_1^2 + x_2^2).$

$$\dot{V} = x_1 x_2 - x_1 x_2 + x_1 x_2 u \quad \Rightarrow \quad u = -L_g V(x) = -x_1 x_2$$

Jurdjevic-Quinn (Nonlinear Damping) Control: If V is such that $L_f V \leq 0$, then $u = -L_g V$ globally asymptotically stabilizes the origin.

Example: $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_1 u \quad$ and $\quad V(x) = \frac{1}{2}(x_1^2 + x_2^2).$

$$\dot{V} = x_1 x_2 - x_1 x_2 + x_1 x_2 u \quad \Rightarrow \quad u = -L_g V(x) = -x_1 x_2$$



V. Jurdjevic and J. P. Quinn, "Controllability and Stability", *J. Diff. Eqs.*, 1978.

Example:

$$\dot{x}_1 = -x_1^3 + x_2 \phi(x_1, x_2),$$
$$\dot{x}_2 = \psi(x_1, x_2) + u.$$

$$V(x_1, x_2) = \tfrac{1}{2}(x_1^2 + x_2^2), \quad u = -x_2 - \phi(x_1, x_2) - \psi(x_1, x_2)x_1 \Rightarrow \dot{V} = -x_1^4 - x_2^2$$

Завдання для самостійного виконання

Визначити функцію керування u=u(x1,x2), що забезпечує стабілізацію нульового розв'язка системи

1. $\dot{x} = x^2 + u.$

2. $\dot{x}_1 = -x_1^3 + x_2,$
$\dot{x}_2 = x_2 + u.$