Ряди

Sums and Products

This constructs the sum $\sum_{i=1}^{7} \frac{x^i}{i}$:

In[1]:= **Sum[x^i/i, {i, 1, 7}]**

Out[1]= $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\frac{x^6}{6}+\frac{x^7}{7}$

You can leave out the lower limit if it is equal to 1:

In[2]:= **Sum[x^i/i, {i, 7}]**

Out[2]= $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\frac{x^6}{6}+\frac{x^7}{7}$

This makes i increase in steps of 2, so that only odd-numbered values are included:

In[3]:= **Sum[x^i/i, {i, 1, 5, 2}]**

Out[3]= $x+\frac{x^3}{3}+\frac{x^5}{5}$

Products work just like sums:

In[4]:= **Product[x + i, {i, 1, 4}]**

Out[4]= $(1+x)(2+x)(3+x)(4+x)$

Sum[f, {i, i_{min}, i_{max}}]	the sum $\sum_{i=i_{min}}^{i_{max}} f$
Sum[f, {i, i_{min}, i_{max}, di}]	the sum with i increasing in steps of di
Sum[f, {i, i_{min}, i_{max}}, {j, j_{min}, j_{max}}]	the nested sum $\sum_{i=i_{min}}^{i_{max}} \sum_{j=j_{min}}^{j_{max}} f$
Product[f, {i, i_{min}, i_{max}}]	the product $\prod_{i=i_{min}}^{i_{max}} f$

Sums and products.

This sum is computed symbolically as a function of n:

SumConvergence

SumConvergence[f, n]
gives conditions for the sum $\sum_n^\infty f$ to be convergent.

SumConvergence[f, {n_1, n_2, …}]
gives conditions for the multiple sum $\sum_{n_1}^\infty \sum_{n_2}^\infty \dots f$ to be convergent.

Details and Options

Examples open all

Basic Examples (2)

Test for convergence of the sum $\sum_n^\infty \frac{1}{n}$:

In[1]:= **SumConvergence[1 / n, n]**

Out[1]= False

Test the convergence of $\sum_n^\infty \frac{3^n n^2}{n!}$:

In[2]:= **SumConvergence[3 ^ n n ^ 2 / n!, n]**

Out[2]= True

Find the condition for convergence of $\sum_n^\infty \frac{1}{n^\alpha}$:

In[1]:= **SumConvergence[1 / n ^ α, n]**

Out[1]= Re[α] > 1

Method (9)

In[1]:= SumConvergence[a^n/n!, n, Method → "RatioTest"]

Out[1]= True

In[1]:= SumConvergence[(2 n + 1)!/((5 n)! n), n, Method → "RatioTest"]

Out[1]= True

In this case the ratio test is inconclusive:

In[1]:= SumConvergence[1/n, n, Method → "RatioTest"]

Out[1]= $\text{SumConvergence}\left[\frac{1}{n}, n, \text{Method} \to \text{RatioTest}\right]$

In[1]:= SumConvergence[x^n/n, n, Method → "RootTest"]

Out[1]= Abs[x] < 1

In[1]:= $\text{SumConvergence}\left[\left(\frac{2 n+3}{5 n-4}\right)^n, n, \text{Method} \to \text{"RootTest"}\right]$

Out[1]= True

In this case the root test is inconclusive:

In[1]:= SumConvergence[(1 – 1/n)^n, n, Method → "RootTest"]

Out[1]= $\text{SumConvergence}\left[\left(1-\frac{1}{n}\right)^n, n, \text{Method} \to \text{RootTest}\right]$

The Raabe test works well for rational functions:

In[1]:= SumConvergence[n / (n ^ 3 + 2 n + 1), n, Method → "RaabeTest"]

Out[1]= True

In[2]:= SumConvergence[1 / Sqrt[n], n, Method → "RaabeTest"]

Out[2]= False

In[3]:= SumConvergence$\left[\frac{\prod_{k=1}^{n} (2 k - 1)}{\prod_{k=1}^{n} 2 k}\right.$, n, Method → "RaabeTest"]

Out[3]= False

In this case the Raabe test is inconclusive:

In[4]:= SumConvergence[x ^ n, n, Method → "RaabeTest"]

Out[4]= SumConvergence[x^n, n, Method → RaabeTest]

In[1]:= SumConvergence[1 / Log[n] ^ 2, n, Method → "IntegralTest"]

Out[1]= False

In[1]:= SumConvergence[1 / (n Log[n] × Log[Log[n]] ^ 2), n, Method → "IntegralTest"]

Out[1]= True

In this case the integral test is inconclusive:

In[1]:= SumConvergence[1 / Prime[n], n, Method → "IntegralTest"]

Out[1]= SumConvergence$\left[\frac{1}{\text{Prime}[n]}\right.$, n, Method → IntegralTest]

Applications

Find the radius of convergence of a power series:

In[1]:= **SumConvergence[x^n, n]**

Out[1]= Abs[x] < 1

In[1]:= **Sum[x^n, {n, 0, Infinity}, GenerateConditions → True]**

Out[1]= $\frac{1}{1-x}$ if Abs[x] < 1

Find the interval of convergence for a real power series:

In[1]:= **SumConvergence[(x^n) / (n 3^n), n]**

Out[1]= Abs[x] < 3 || x == −3

As a real power series, this converges on the interval [-3, 3):

In[2]:= **SumConvergence[(x^n) / (n 3^n), n, Assumptions → x ∈ Reals]**

Out[2]= −3 ≤ x < 3

Prove convergence of Ramanujan's formula for $\frac{1}{\pi}$:

In[1]:= **SumConvergence[Sqrt[8] / 9801 (4 n)! (1103 + 26 390 n) / (n!)^4 / 396^(4 n), n]**

Out[1]= True

Sum it:

In[2]:= **Sqrt[8] / 9801 × Sum[(4 n)! (1103 + 26 390 n) / (n!)^4 / 396^(4 n), {n, 0, Infinity}]**

Out[2]= $\frac{1}{\pi}$

› Properties & Relations (4)

› Neat Examples (1)

Power Series

The mathematical operations we have discussed so far are *exact*. Given precise input, their results are exact formulas.

In many situations, however, you do not need an exact result. It may be quite sufficient, for example, to find an *approximate* formula that is valid, say, when the quantity x is small.

This gives a power series approximation to $(1+x)^n$ for x close to 0, up to terms of order x^3:

In[1]:=
```
Series[(1 + x)^n, {x, 0, 3}]
```

Out[1]=

$$1+n\,x+\frac{1}{2}(-1+n)\,n\,x^2+\frac{1}{6}(-2+n)(-1+n)\,n\,x^3+O[x]^4$$

The Wolfram Language knows the power series expansions for many mathematical functions:

In[2]:=
```
Series[Exp[-a t] (1 + Sin[2 t]), {t, 0, 4}]
```

Out[2]=

$$1+(2-a)\,t+\left(-2\,a+\frac{a^2}{2}\right)t^2+\left(-\frac{4}{3}+a^2-\frac{a^3}{6}\right)t^3+\frac{1}{24}\left(32\,a-8\,a^3+a^4\right)t^4+O[t]^5$$

If you give it a function that it does not know, Series writes out the power series in terms of derivatives:

In[3]:=
```
Series[1 + f[t], {t, 0, 3}]
```

Out[3]=

$$(1+f[0])+f'[0]\,t+\frac{1}{2}f''[0]\,t^2+\frac{1}{6}f^{(3)}[0]\,t^3+O[t]^4$$

Power series are approximate formulas that play much the same role with respect to algebraic expressions as approximate numbers play with respect to numerical expressions. The Wolfram Language allows you to perform operations on power series, in all cases maintaining the appropriate order or "degree of precision" for the resulting power series.

Here is a simple power series, accurate to order x^5:

In[4]:=
```
Series[Exp[x], {x, 0, 5}]
```

Out[4]=

$$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O[x]^6$$

When you do operations on a power series, the result is computed only to the appropriate order in x:

In[5]:=
```
%^2 (1 + %)
```

In[4]:= **Series[Exp[x], {x, 0, 5}]**

Out[4]= $1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O[x]^6$

When you do operations on a power series, the result is computed only to the appropriate order in x:

In[5]:= **%^2 (1 + %)**

Out[5]= $2+5x+\frac{13x^2}{2}+\frac{35x^3}{6}+\frac{97x^4}{24}+\frac{55x^5}{24}+O[x]^6$

This turns the power series back into an ordinary expression:

In[6]:= **Normal[%]**

Out[6]= $2+5x+\frac{13x^2}{2}+\frac{35x^3}{6}+\frac{97x^4}{24}+\frac{55x^5}{24}$

Now the square is computed *exactly*:

In[7]:= **%^2**

Out[7]= $\left(2+5x+\frac{13x^2}{2}+\frac{35x^3}{6}+\frac{97x^4}{24}+\frac{55x^5}{24}\right)^2$

Applying Expand gives a result with 11 terms:

In[8]:= **Expand[%]**

Out[8]= $4+20x+51x^2+\frac{265x^3}{3}+\frac{467x^4}{4}+\frac{1505x^5}{12}+\frac{7883x^6}{72}+\frac{1385x^7}{18}+\frac{24809x^8}{576}+\frac{5335x^9}{288}+\frac{3025x^{10}}{576}$

Series[*expr*, {x, x_0, n}]	find the power series expansion of *expr* about the point $x=x_0$ to at most n^{th} order
Normal[*series*]	truncate a power series to give an ordinary expression

Power series operations.

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Making Power Series Expansions

Making Power Series Expansions

Series[$expr$, {x, x_0, n}]	find the power series expansion of $expr$ about the point $x=x_0$ to order at most $(x-x_0)^n$
Series[$expr$, {x, x_0, n_x}, {y, y_0, n_y}]	find series expansions with respect to y, then x

Functions for creating power series.

Here is the power series expansion for $\exp(x)$ about the point $x = 0$ to order x^4:

In[1]:= **Series[Exp[x], {x, 0, 4}]**

Out[1]= $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + O[x]^5$

Here is the series expansion of $\exp(x)$ about the point $x = 1$:

In[2]:= **Series[Exp[x], {x, 1, 4}]**

Out[2]= $e + e(x-1) + \frac{1}{2} e (x-1)^2 + \frac{1}{6} e (x-1)^3 + \frac{1}{24} e (x-1)^4 + O[x-1]^5$

If the Wolfram Language does not know the series expansion of a particular function, it writes the result symbolically in terms of derivatives:

In[3]:= **Series[f[x], {x, 0, 3}]**

Out[3]= $f[0] + f'[0]\, x + \frac{1}{2} f''[0]\, x^2 + \frac{1}{6} f^{(3)}[0]\, x^3 + O[x]^4$

In mathematical terms, Series can be viewed as a way of constructing Taylor series for functions.

The standard formula for the Taylor series expansion about the point $x = x_0$ of a function $g(x)$ with k th derivative $g^{(k)}(x)$ is $g(x) = \sum_{k=0}^{\infty} g^{(k)}(x_0) \frac{(x-x_0)^k}{k!}$. Whenever this formula applies, it gives the same results as Series. (For common functions, Series nevertheless internally uses somewhat more efficient algorithms.)

Series can also generate some power series that involve fractional and negative powers, not directly covered by the standard Taylor series formula.

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Basic Examples (4)

Find the coefficient for a term in a series:

In[1]:= **Series[Exp[Sin[x]], {x, 0, 10}]**

Out[1]= $1+x+\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^5}{15}-\frac{x^6}{240}+\frac{x^7}{90}+\frac{31\,x^8}{5760}+\frac{x^9}{5670}-\frac{2951\,x^{10}}{3628800}+O[x]^{11}$

In[2]:= **SeriesCoefficient[%, 8]**

Out[2]= $\frac{31}{5760}$

Find the coefficient of the general term in a series:

In[1]:= **SeriesCoefficient[Exp[−x], {x, 0, n}]**

Out[1]= $\begin{cases} \frac{(-1)^n}{n!} & n \geq 0 \\ 0 & \text{True} \end{cases}$

In[2]:= **Table[%, {n, 0, 5}]**

Out[2]= $\left\{1, -1, \frac{1}{2}, -\frac{1}{6}, \frac{1}{24}, -\frac{1}{120}\right\}$

In[3]:= **Series[Exp[−x], {x, 0, 5}]**

Out[3]= $1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+O[x]^6$

Find the coefficient for a term in a multivariate series:

In[1]:= **Series[Exp[x + y], {x, 0, 3}, {y, 0, 3}]**

Out[1]= $\left(1+y+\frac{y^2}{2}+\frac{y^3}{6}+O[y]^4\right)+\left(1+y+\frac{y^2}{2}+\frac{y^3}{6}+O[y]^4\right)x+\left(\frac{1}{2}+\frac{y}{2}+\frac{y^2}{4}+\frac{y^3}{12}+O[y]^4\right)x^2+\left(\frac{1}{6}+\frac{y}{6}+\frac{y^2}{12}+\frac{y^3}{36}+O[y]^4\right)x^3+O[x]^4$

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Examples

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Basic Examples (2)

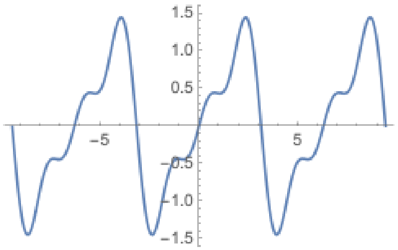
Find the 3^{rd}-order Fourier series of $\frac{t}{2}$:

In[1]:= `FourierSeries[t/2, t, 3]`

Out[1]= $\frac{1}{2} i e^{-i t} - \frac{1}{2} i e^{i t} - \frac{1}{4} i e^{-2 i t} + \frac{1}{4} i e^{2 i t} + \frac{1}{6} i e^{-3 i t} - \frac{1}{6} i e^{3 i t}$

In[2]:= `Plot[%, {t, -3 Pi, 3 Pi}]`

Out[2]=



Compute an order $\{2, 2\}$ Fourier series:

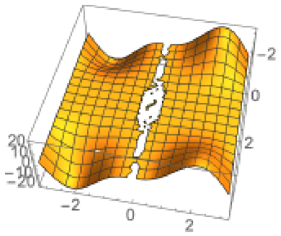
In[1]:= `FourierSeries[x^2 y, {x, y}, {2, 2}]`

Out[1]=

$$-\frac{1}{4} i e^{i(-2x-2y)} + i e^{i(-x-2y)} + i e^{i(x-2y)} - \frac{1}{4} i e^{i(2x-2y)} + \frac{1}{2} i e^{i(-2x-y)} - 2 i e^{i(-x-y)} - 2 i e^{i(x-y)} + \frac{1}{2} i e^{i(2x-y)} - \frac{1}{2} i e^{i(-2x+y)} + 2 i e^{i(-x+y)} + 2 i e^{i(x+y)} - \frac{1}{2} i e^{i(2x+y)} + \frac{1}{4} i e^{i(-2x+2y)} - i e^{i(-x+2y)} - i e^{i(x+2y)} + \frac{1}{4} i e^{i(2x+2y)} + \frac{1}{3} i e^{-i y} \pi^2 - \frac{1}{3} i e^{i y} \pi^2 - \frac{1}{6} i e^{-2 i y} \pi^2 + \frac{1}{6} i e^{2 i y} \pi^2$$

In[2]:= `Plot3D[%, {x, -Pi, Pi}, {y, -Pi, Pi}]`

Out[2]=



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Examples

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Basic Examples (4)

Find the coefficient for a term in a series:

In[1]:= **Series[Exp[Sin[x]], {x, 0, 10}]**

Out[1]= $1+x+\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^5}{15}-\frac{x^6}{240}+\frac{x^7}{90}+\frac{31\,x^8}{5760}+\frac{x^9}{5670}-\frac{2951\,x^{10}}{3628800}+O[x]^{11}$

In[2]:= **SeriesCoefficient[%, 8]**

Out[2]= $\frac{31}{5760}$

Find the coefficient of the general term in a series:

In[1]:= **SeriesCoefficient[Exp[−x], {x, 0, n}]**

Out[1]= $\begin{cases} \frac{(-1)^n}{n!} & n \geq 0 \\ 0 & \text{True} \end{cases}$

In[2]:= **Table[%, {n, 0, 5}]**

Out[2]= $\left\{1, -1, \frac{1}{2}, -\frac{1}{6}, \frac{1}{24}, -\frac{1}{120}\right\}$

In[3]:= **Series[Exp[−x], {x, 0, 5}]**

Out[3]= $1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+O[x]^6$

Find the coefficient for a term in a multivariate series:

In[1]:= **Series[Exp[x + y], {x, 0, 3}, {y, 0, 3}]**

Out[1]= $\left(1+y+\frac{y^2}{2}+\frac{y^3}{6}+O[y]^4\right)+\left(1+y+\frac{y^2}{2}+\frac{y^3}{6}+O[y]^4\right)x+\left(\frac{1}{2}+\frac{y}{2}+\frac{y^2}{4}+\frac{y^3}{12}+O[y]^4\right)x^2+\left(\frac{1}{6}+\frac{y}{6}+\frac{y^2}{12}+\frac{y^3}{36}+O[y]^4\right)x^3+O[x]^4$

FourierTrigSeries [*expr*, *t*, *n*]
gives the n^{th}-order Fourier trigonometric series expansion of *expr* in *t*.

FourierTrigSeries [*expr*, { t_1, t_2, …}, { n_1, n_2, …}]
gives the multidimensional Fourier trigonometric series of *expr*.

Details and Options

Examples

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Basic Examples (2)

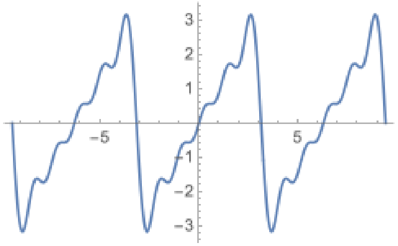
Find the 5^{th}-order Fourier trigonometric series of *t*:

In[1]:= **FourierTrigSeries[t, t, 5]**

Out[1]= $2\,\text{Sin}[t] - \text{Sin}[2\,t] + \frac{2}{3}\,\text{Sin}[3\,t] - \frac{1}{2}\,\text{Sin}[4\,t] + \frac{2}{5}\,\text{Sin}[5\,t]$

In[2]:= **Plot[%, {t, −3 π, 3 π}]**

Out[2]=

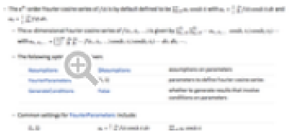


Find the 3rd-order bivariate Fourier trigonometric series approximation to

FourierCosSeries [*expr*, *t*, *n*]
gives the n^{th}-order Fourier cosine series expansion of *expr* in *t*.

FourierCosSeries [*expr*, { t_1, t_2, … } , { n_1, n_2, … }]
gives the multidimensional Fourier cosine series of *expr*.

Details and Options



Examples

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Basic Examples (2)

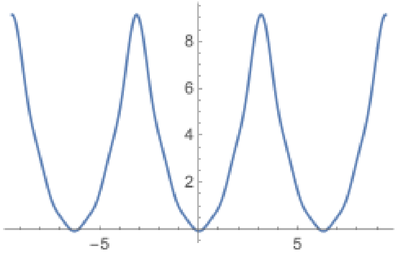
Find the 5^{th}-order Fourier cosine series of t^2:

In[1]:=
```
FourierCosSeries[t^2, t, 5]
```

Out[1]=
$$\frac{\pi^2}{3} + 4\left(-\text{Cos}[t] + \frac{1}{4}\text{Cos}[2\,t] - \frac{1}{9}\text{Cos}[3\,t] + \frac{1}{16}\text{Cos}[4\,t] - \frac{1}{25}\text{Cos}[5\,t]\right)$$

In[2]:=
```
Plot[%, {t, -3 Pi, 3 Pi}]
```

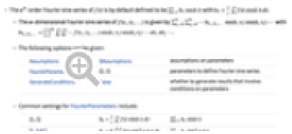
Out[2]=



FourierSinSeries[*expr*, *t*, *n*]
gives the n^{th}-order Fourier sine series expansion of *expr* in *t*.

FourierSinSeries[*expr*, { t_1, t_2, …}, { n_1, n_2, …}]
gives the multidimensional Fourier sine series of *expr*.

Details and Options



Examples

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Basic Examples (2)

Find the 5^{th}-order Fourier sine series approximation to *t*:

In[1]:=
```
FourierSinSeries[t, t, 5]
```

Out[1]= $-2\left(-\text{Sin}[t]+\frac{1}{2}\text{Sin}[2\,t]-\frac{1}{3}\text{Sin}[3\,t]+\frac{1}{4}\text{Sin}[4\,t]-\frac{1}{5}\text{Sin}[5\,t]\right)$

In[2]:=
```
Plot[%, {t, -3 π, 3 π}]
```

Out[2]=