

## РОЗВ'ЯЗАННЯ СИСТЕМ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

$$1) \begin{cases} \dot{x} = 2x + y \\ \dot{y} = 3x + 4y \end{cases} \quad \dot{x} \approx \frac{dx}{dt}, \quad \dot{y} \approx \frac{dy}{dt}$$

$$\bullet \quad \ddot{x} = 2\dot{x} + \dot{y} = 4x + 2y + 3x + 4y = 7x + 6y$$

Утворимо нову систему:

$$\begin{cases} \dot{x} = 2x + y \\ \ddot{x} = 7x + 6y \end{cases} \Rightarrow y = \dot{x} - 2x$$

$$\ddot{x} = 7x + 6\dot{x} - 12x \Rightarrow \ddot{x} - 6\dot{x} + 5x = 0$$

$$k^2 - 6k + 5 = 0 \Rightarrow k_{1,2} = 3 \pm \sqrt{9-5} = 3 \pm 2, \quad \Rightarrow k_{1,2} = 1; 5$$

$$x(t) = C_1 e^t + C_2 e^{5t}$$

$$y(t) = \dot{x} - 2x = C_1 e^t + 5C_2 e^{5t} - 2C_1 e^t - 2C_2 e^{5t} = -C_1 e^t + 3C_2 e^{5t}$$

Загальний розв'язок має вигляд:

$$\begin{cases} x(t) = C_1 e^t + C_2 e^{5t} \\ y(t) = -C_1 e^t + 3C_2 e^{5t} \end{cases} \bullet$$

$$2) \begin{cases} \dot{x} = 3x - 4y \\ \dot{y} = x - 2y \end{cases}$$

$$\bullet \quad x = \dot{y} + 2y$$

$$\ddot{y} = \dot{x} - 2\dot{y} = 3x - 4y - 2x + 4y = \dot{y} + 2y \Rightarrow \ddot{y} - \dot{y} - 2y = 0$$

$$k^2 - k - 2 = 0 \Rightarrow k_{1,2} = \frac{1 \pm \sqrt{1+8}}{2}; \Rightarrow k_1 = -1; k_2 = 2$$

$$y = C_1 e^{-t} + C_2 e^{2t}$$

$$\underline{x = \dot{y} + 2y = -C_1 e^{-t} + 2C_2 e^{2t} + 2C_1 e^{-t} + 2C_2 e^{2t} = C_1 e^{-t} + 4C_2 e^{2t}} \bullet$$

$$3) \begin{cases} 2\dot{x} - 5\dot{y} = -x + 4y \\ 3\dot{x} - 4\dot{y} = 2x - y \end{cases} \begin{array}{l|l} 4 & 3 \\ + & + \\ -5 & -2 \end{array}$$

$$\bullet \begin{cases} -7\dot{x} = 21y - 14x \\ -7\dot{y} = 14y - 7x \end{cases} \Rightarrow \begin{cases} \dot{x} = 2x - 3y \\ \dot{y} = x - 2y \end{cases}$$

$$\ddot{x} = 2\dot{x} - 3\dot{y} = 4x - 6y - 3x + 6y \Rightarrow \ddot{x} = x$$

$$k^2 - 1 = 0 \Rightarrow k_{1,2} = \pm 1$$

$$\underline{x(t) = C_1 e^{-t} + C_2 e^t}$$

$$\underline{y(t) = \frac{2x - \dot{x}}{3} = \frac{2C_1 e^{-t} + 2C_2 e^t + C_1 e^{-t} - C_2 e^t}{3} = C_1 e^{-t} + \frac{1}{3} C_2 e^t} \bullet$$

$$4) \begin{cases} \dot{x} + x - 8y = 0 \\ \dot{y} - x - y = 0 \end{cases}$$

$$\bullet \begin{cases} \dot{x} = -x + 8y \\ \dot{y} = x + y \end{cases}$$

$$\ddot{x} + \dot{x} - 8\dot{y} = 0$$

$$\ddot{x} - x + 8y - 8x - 8y = 0 \quad \ddot{x} - 9x = 0$$

$$k^2 - 9 = 0 \Rightarrow k_{1,2} = \pm 3$$

$$\underline{x = C_1 e^{-3t} + C_2 e^{3t}}$$

$$\underline{y = \frac{\dot{x} + x}{8} = \frac{-3C_1 e^{-3t} + 3C_2 e^{3t} + C_1 e^{-3t} + C_2 e^{3t}}{8} = \frac{-2C_1 e^{-3t} + 4C_2 e^{3t}}{8} =$$

$$\underline{= -\frac{1}{4} C_1 e^{-3t} + \frac{1}{2} C_2 e^{3t}} \bullet$$

$$3) \begin{cases} \dot{x} = y + 2e^t \\ \dot{y} = x + t^2 \end{cases}$$

$$\bullet \quad \ddot{x} = \dot{y} + 2e^t = x + t^2 + 2e^t \quad \Rightarrow \quad \ddot{x} - x = t^2 + 2e^t;$$

$$k^2 - 1 = 0 \quad \Rightarrow \quad k_{1,2} = \pm 1; \quad \underline{x_0 = C_1 e^{-t} + C_2 e^t};$$

$$\ddot{x} - x = t^2$$

$$\bar{x}_1 = At^2 + Bt + C \quad \Rightarrow \quad \bar{x}'_1 = 2At + B \quad \Rightarrow \quad \bar{x}''_1 = 2A$$

$$2A - At^2 - Bt - C = t^2$$

$$\begin{array}{l|l} t^2 & -A = 1 \\ t & -B = 0 \\ t^0 & 2A - C = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} A = -1 \\ B = 0 \\ C = -2 \end{array} \quad \underline{\bar{x}_1 = -2 - t^2};$$

$$\ddot{x} - x = 2e^t$$

$$\bar{x}_2 = Dte^t \quad \Rightarrow \quad \bar{x}'_2 = Dte^t + De^t \quad \Rightarrow \quad \bar{x}''_2 = Dte^t + 2De^t$$

$$Dte^t + 2De^t - Dte^t = 2e^t \quad \Rightarrow \quad 2De^t = 2e^t \quad \Rightarrow \quad D = 1; \quad \underline{\bar{x}_2 = te^t};$$

$$x = C_1 e^{-t} + C_2 e^t - 2 - t^2 + te^t;$$

$$\begin{cases} x = C_1 e^{-t} + C_2 e^t - 2 - t^2 + te^t \\ y = -C_1 e^{-t} + C_2 e^t - 2t + te^t - e^t \end{cases} \bullet$$

$$4) \begin{cases} \dot{x} = 3x + 2y + 4e^{5t} \\ \dot{y} = x + 2y \end{cases}$$

$$\bullet \quad x = \dot{y} - 2y$$

$$\ddot{y} = \dot{x} + 2\dot{y} = 3x + 2y + 4e^{5t} + 2x + 4y = 6y + 4e^{5t} + 5\dot{y} - 10y$$

$$\ddot{y} - 5\dot{y} + 4y = 4e^{5t}$$

$$k^2 - 5k + 4 = 0 \quad \Rightarrow \quad k_{1,2} = \frac{5 \pm \sqrt{25-16}}{2} = \frac{5 \pm 3}{2}; \Rightarrow k_1 = 1; k_2 = 4$$

$$y_o = C_1 e^t + C_2 e^{4t} \quad \bar{y} = A e^{5t}$$

$$25Ae^{5t} - 25Ae^{5t} + 4Ae^{5t} = 4e^{5t} \Rightarrow A = 1$$

$$\underline{y = C_1 e^t + C_2 e^{4t} + e^{5t}}$$

$$\underline{x = \dot{y} - 2y = C_1 e^t + 4C_2 e^{4t} + 5e^{5t} - 2C_1 e^t - 2C_2 e^{4t} - 2e^{5t} =}$$

$$\underline{= -C_1 e^t + 2C_2 e^{4t} + 3e^{5t}} \bullet$$

$$5) \begin{cases} \dot{x} = x + z - y \\ \dot{y} = x + y - z \\ \dot{z} = 2x - y \end{cases}$$

$$\bullet \ddot{z} = 2\dot{x} - \dot{y} = 2x + 2z - 2y - x - y + z = x + 3z - 3y$$

$$\ddot{z} = \dot{x} + 3\dot{z} - 3\dot{y} = x + z - y + 6x - 3y - 3x - 3y + 3z = 4x + 4z - 7y$$

Утворимо нову систему:

$$\begin{cases} \dot{z} = 2x - y \\ \ddot{z} = x + 3z - 3y & y = 2x - \dot{z} \\ \ddot{z} = 4x + 4z - 7y \end{cases}$$

$$\ddot{z} = x + 3z - 6x + 3\dot{z} \Rightarrow -5x = \ddot{z} - 3z - 3\dot{z}$$

$$\ddot{z} = 4x + 4z - 14x + 7\dot{z} \Rightarrow \ddot{z} = -10x + 4z + 7\dot{z}$$

$$\ddot{z} = 2\ddot{z} - 6z - 6\dot{z} + 4z + 7\dot{z} = 2\ddot{z} + \dot{z} - 2z$$

$$\ddot{z} - 2\ddot{z} - \dot{z} + 2z = 0$$

$$k^3 - 2k^2 - k + 2 = 0 \Rightarrow k^2(k-2) - (k-2) = 0 \Rightarrow (k^2-1)(k-2) = 0 \Rightarrow$$

$$\Rightarrow k_1 = 2, k_{2,3} = \pm 1$$

$$\underline{z = C_1 e^{-t} + C_2 e^t + C_3 e^{2t}}$$

$$\underline{x = -\frac{1}{5}(\ddot{z} - 3z - 3\dot{z}) =}$$

$$= \frac{1}{5}(-C_1 e^{-t} - C_2 e^t - 4C_3 e^{2t} + 3C_1 e^{-t} + 3C_2 e^t + 3C_3 e^{2t} - 3C_1 e^{-t} + 3C_2 e^t + 6C_3 e^{2t}) =$$

$$\underline{= \frac{-1}{5}C_1 e^{-t} + C_2 e^t + C_3 e^{2t}}$$

$$\begin{aligned} \underline{y} = 2x - \dot{z} &= \frac{-2}{5}C_1e^{-t} + 2C_2e^t + 2C_3e^{2t} + C_1e^{-t} - C_2e^t - 2C_3e^{2t} = \\ &= \underline{3/5 \cdot C_1e^{-t} + C_2e^t} \bullet \end{aligned}$$

$$\mathbf{6) \begin{cases} \dot{x} = 2x - y + z \\ \dot{y} = x + 2y - z \\ \dot{z} = x - y + 2z \end{cases}}$$

$$\begin{aligned} \bullet \\ \ddot{x} = 2\dot{x} - \dot{y} + \dot{z} &= 4x - 2y + 2z - x - 2y + z + x - y + 2z = \\ &= 4x - 5y + 5z = 4x + 5(-y + z); \end{aligned}$$

$$\left. \begin{aligned} \ddot{x} &= 4x + 5(-y + z) \\ -y + z &= \dot{x} - 2x \end{aligned} \right\} \Rightarrow \ddot{x} = 4x + 5\dot{x} - 10x;$$

$$\ddot{x} - 5\dot{x} + 6x = 0$$

$$k^2 - 5k + 6 = 0$$

$$k_{1,2} = \frac{5 \pm \sqrt{25 - 24}}{2} = \frac{5 \pm 1}{2} \Rightarrow k_1 = 2, k_2 = 3; \quad \underline{x(t) = C_1e^{2t} + C_2e^{3t}}$$

$$(1)+(2): \quad \dot{x} + \dot{y} = 3x + y \Rightarrow \dot{y} - y = 3x - \dot{x} \Rightarrow$$

$$\Rightarrow \dot{y} - y = 3C_1e^{2t} + 3C_2e^{3t} - 2C_1e^{2t} - 3C_2e^{3t} \Rightarrow \dot{y} - y = C_1e^{2t}$$

$$y(t) = y_0(t) + \bar{y}(t); \quad k - 1 = 0 \Rightarrow k = 1; \quad y_0(t) = C_3e^t$$

$$\bar{y}(t) = Ae^{2t}; \quad 2Ae^{2t} - Ae^{2t} = C_1e^{2t} \Rightarrow A = C_1; \quad \bar{y}(t) = C_1e^{2t}$$

$$\underline{y(t) = C_1e^{2t} + C_3e^t}$$

$$\begin{aligned} \underline{z(t)} &= \dot{x} - 2x + y = 2C_1e^{2t} + 3C_2e^{3t} - 2C_1e^{2t} - 2C_2e^{3t} + C_1e^{2t} + C_3e^t = \\ &= \underline{C_1e^{2t} + C_2e^{3t} + C_3e^t} \bullet \end{aligned}$$

### Домашнє завдання.

$$\mathbf{1) \begin{cases} \dot{x} = y - 5\cos t \\ \dot{y} = 2x + y \end{cases}}$$

$$\mathbf{2) \begin{cases} \dot{x} = 2x - y + 2z \\ \dot{y} = x + 2z \\ \dot{z} = -2x + y - z \end{cases}}$$