ЛІНІЙНІ НЕОДНОРІДНІ РІВНЯННЯ
З ПОСТІЙНИМИ КОЕФІЦІЄНТАМИ

1) $y'' - 9y = e^{3x}\cos x$.

- $k^2 - 9 = 0 \quad \Rightarrow \quad k_1 = -3,\ \ k_2 = 3; \qquad y(x) = C_1 e^{-3x} + C_2 e^{3x}$.

$\bar{y} = e^{3x}(A\cos x + B\sin x)$;

$\bar{y}' = 3e^{3x}(A\cos x + B\sin x) + e^{3x}(-A\sin x + B\cos x)$;

$\underline{9e^{3x}(A\cos x + B\sin x)} + 6e^{3x}(-A\sin x + B\cos x) - e^{3x}(A\cos x + B\sin x) -$

$-\underline{9e^{3x}(A\cos x + B\sin x)} = e^{3x}\cos x$;

$$\left.\begin{array}{l|l} e^{3x}\sin x & -6A - B = 0 \\ e^{3x}\cos x & 6B - A = 1 \end{array}\right\} \quad \Rightarrow \quad \begin{array}{l} B = -6A \\ A = -\dfrac{1}{37} \end{array} \quad \Rightarrow \quad B = \frac{6}{37};$$

$$\bar{y} = e^{3x}\left(-\frac{1}{37}\cos x + \frac{6}{37}\sin x\right);$$

$$y(x) = C_1 e^{-3x} + C_2 e^{3x} + e^{3x}\left(-\frac{1}{37}\cos x + \frac{6}{37}\sin x\right). \ \bullet$$

2) $y'' + y = \cos 2x + \cos x$.

- $k^2 + 1 = 0 \quad \Rightarrow \quad k_{1,2} = \pm i; \qquad y_o(x) = C_1\cos x + C_2\sin x$.

В правій частині маємо доданки різних типів. Тому знаходження частинного розвязку зручно розбити на два етапи: $\bar{y} = \bar{y}_1 + \bar{y}_2$.

а) $y'' + y = \cos 2x$

$\bar{y}_1 = A\cos 2x + B\sin 2x$;

$-4A\cos 2x - 4B\sin 2x + A\cos 2x + B\sin 2x = \cos 2x$;

$$\left.\begin{array}{l|l} \cos 2x & -4A + A = 1 \\ \sin 2x & -4B + B = 0 \end{array}\right\} \quad \Rightarrow \quad \begin{array}{c} A = -\dfrac{1}{3} \\ B = 0 \end{array} \qquad \bar{y}_1 = -\frac{1}{3}\cos 2x;$$

б) $y'' + y = \cos x$

$\bar{y}_2 = x(C\cos x + D\sin x) \ \Rightarrow \ \bar{y}_2' = (C\cos x + D\sin x) + x(-C\sin x + D\cos x)$;

$2(-C\sin x + D\cos x) \underline{- x(C\cos x + D\sin x)} + \underline{x(C\cos x + D\sin x)} = \cos x$;

$$\left.\begin{array}{l|l} \cos x & 2D = 1 \\ \sin x & -2C = 0 \end{array}\right\} \Rightarrow \begin{array}{l} D = \frac{1}{2} \\ C = 0 \end{array} \qquad \bar{y}_2 = \frac{x}{2}\sin x.$$

Тоді $\qquad y(x) = C_1\cos x + C_2\sin x - \frac{1}{3}\cos 2x + \frac{x}{2}\sin x.$ •

3) $y'' - 5y' = 3x^2 + \sin 5x$.

• $k^2 - 5k = 0 \quad \Rightarrow \quad k_1 = 0, \ k_2 = 5; \qquad y(x) = C_1 + C_2 e^{5x}$.

$\bar{y} = \bar{y}_1 + \bar{y}_2$;

а) $\bar{y}_1 = \left(Ax^2 + Bx + C\right)x = Ax^3 + Bx^2 + Cx \quad \Rightarrow$

$\Rightarrow \quad \bar{y}_1' = 3Ax^2 + 2Bx + C \quad \Rightarrow \quad \bar{y}_1'' = 6Ax + 2B$;

$6Ax + 2B - 15Ax^2 - 10Bx - 5C = 3x^2$

$$\left.\begin{array}{r|c} x^2 & -15A = 3 \\ x & 6A - 10B = 0 \\ x^0 & 2B - 5C = 0 \end{array}\right\} \Rightarrow \begin{array}{l} A = -\frac{1}{5} \\ B = -\frac{3}{25} \\ C = -\frac{6}{125} \end{array} \qquad \bar{y}_1 = -\frac{1}{5}x^3 - \frac{3}{25}x^2 - \frac{6}{125}x;$$

б) $\bar{y}_2 = D\sin 5x + K\cos 5x$

$-25D\sin 5x - 25K\cos 5x - 25D\cos 5x + 25K\sin 5x = \sin 5x$;

$$\left.\begin{array}{l|l} \sin 5x & -25D + 25K = 1 \\ \cos 5x & -25K - 25D = 0 \end{array}\right\} \Rightarrow \begin{array}{l} D = -\frac{1}{50} \\ K = \frac{1}{50} \end{array}$$

$$\bar{y}_2(x) = -\frac{1}{50}\sin 5x + \frac{1}{50}\cos 5x;$$

$$y(x) = C_1 + C_2 e^{5x} - \frac{1}{5}x^3 - \frac{3}{25}x^2 - \frac{6}{125}x - \frac{1}{50}\sin 5x + \frac{1}{50}\cos 5x. \quad \bullet$$

4) $y'' - 4y' + 8y = e^{2x} + \sin 2x$

• $k^2 - 4k + 8 = 0 \;\Rightarrow\quad k_{1,2} = 2 \pm \sqrt{4-8} = 2 \pm 2i;$

$y_o(x) = e^{2x}(C_1 \cos 2x + C_2 \sin 2x);\qquad \bar{y} = \bar{y}_1 + \bar{y}_2;$

а) $y'' - 4y' + 8y = e^{2x};$

$\bar{y}_1 = Ae^{2x};$

$$4Ae^{2x} - \underline{8Ae^{2x}} + \underline{8Ae^{2x}} = e^{2x} \quad\Rightarrow\quad 4A = 1 \Rightarrow \quad A = \frac{1}{4};\qquad \bar{y}_1 = \frac{1}{4}e^{2x};$$

б) $y'' - 4y' + 8y = \sin 2x$

$\bar{y}_2 = B\sin 2x + C\cos 2x;$

$$-4B\sin 2x - 4C\cos 2x - 4(2B\cos 2x - 2C\sin 2x) + 8(B\sin 2x + C\cos 2x) = \sin 2x;$$

$$\begin{array}{l|l} \sin 2x & -4B + 8C + 8B = 1 \\ \cos 2x & -4C - 8B + 8C = 0 \end{array}\Bigg\} \quad\Rightarrow\quad \begin{array}{l} 4B + 8C = 1 \\ 4C - 8B = 0 \end{array}\Bigg\} \qquad \begin{array}{l} B = \dfrac{1}{20} \\ C = \dfrac{1}{10} \end{array}$$

$$\bar{y}_2(x) = \frac{1}{20}\sin 2x + \frac{1}{10}\cos 2x;$$

$$y(x) = e^{2x}(C_1 \cos 2x + C_2 \sin 2x) + \frac{1}{4}e^{2x} + \frac{1}{20}\sin 2x + \frac{1}{10}\cos 2x. \quad \bullet$$

5) $y'' + 6y' + 10y = 3xe^{-3x} - 2e^{3x}\cos x$

• $k^2 + 6k + 10 = 0 \quad\Rightarrow\quad k_{1,2} = -3 \pm \sqrt{9-10} = -3 \pm i;$

$y_o(x) = e^{-3x}(C_1 \cos x + C_2 \sin x);\qquad \bar{y} = \bar{y}_1 + \bar{y}_2;$

а) $y'' + 6y' + 10y = 3xe^{-3x}$

$\bar{y}_1 = (Ax + B)e^{-3x} \quad \Rightarrow \quad \bar{y}_1' = Ae^{-3x} - 3(Ax + B)e^{-3x}$;

$$-\underline{3Ae^{-3x}} - \underline{3Ae^{-3x}} + 9(Ax + B)e^{-3x} + \underline{6Ae^{-3x}} - 18(Ax + B)e^{-3x} +$$

$$+10(Ax + B)e^{-3x} = 3e^{-3x}$$

$$\begin{array}{l|l} e^{-3x} & 9B - 18B + 10B = 0 \\ xe^{-3x} & 9A - 18A + 10A = 3 \end{array}\Bigg\} \quad \Rightarrow \quad \begin{array}{l} B = 0 \\ A = 3 \end{array} \qquad \bar{y}_1 = 3xe^{-3x};$$

б) $y'' + 6y' + 10y = -2e^{3x}\cos x$

$\bar{y}_2 = e^{3x}(C\cos x + D\sin x)$;

$\bar{y}_2' = 3e^{3x}(C\cos x + D\sin x) + e^{3x}(-C\sin x + D\cos x)$;

$$9e^{3x}(C\cos x + D\sin x) + 6e^{3x}(-C\sin x + D\cos x) + e^{3x}(-C\cos x - D\sin x) +$$

$$+18e^{3x}(C\cos x + D\sin x) + 6e^{3x}(-C\sin x + D\cos x) + 10e^{3x}(C\cos x + D\sin x) =$$

$$= -2e^{3x}\cos x;$$

$$\begin{array}{l|l} e^{3x}\cos x & 9C + 6D - C + 18C + 6D + 10C = -2 \\ e^{3x}\sin x & 9D - 6C - D + 18D - 6C + 10D = 0 \end{array}\Bigg\} \quad \Rightarrow$$

$$\Rightarrow \begin{array}{r} 36C + 12D = -2 \\ -12C + 36D = 0 \end{array}\Bigg\} \begin{array}{|l} \cdot 12 \\ \cdot 36 \end{array} \quad \Rightarrow \quad D = -\frac{1}{60}; \quad C = \frac{36D}{12} = -\frac{1}{20};$$

$$\bar{y}_2 = e^{3x}\left(-\frac{1}{20}\cos x - \frac{1}{60}\sin x\right);$$

$$y(x) = e^{-3x}(C_1\cos x + C_2\sin x) + 3xe^{-3x} + e^{3x}\left(-\frac{1}{20}\cos x - \frac{1}{60}\sin x\right). \quad \bullet$$

Домашнє завдання.

1) $y'' - 2y' + 2y = xe^{x} \sin x$;

2) $y'' - 2y' + y = e^{x}(\cos x + 3)$;

3) $y^{IV} - 2y'' + y = 9x^{3} + e^{x}$.