1. $f(x+y) = f(x) + f(y)$, $x, y \in \mathbb{R}$

2. $f(x+y) = f(x) \cdot f(y)$, $x, y \in \mathbb{R}$

3. $f(x \cdot y) = f(x) + f(y)$, $x, y > 0$

4. $f(x \cdot y) = f(x) \cdot f(y)$, $x, y > 0$

$f(x+y) = f(x) + f(y)$

$x = 0 = y$: $f(0) = 2f(0) \Rightarrow \underline{f(0) = 0}$

$-x = y$: $\underbrace{f(0)}_{0} = f(x) + f(-x) \Rightarrow$

$\Rightarrow \underline{f(-x) = -f(x)}$ непарна

$f(x_1 + x_2 + \dots + x_n) = f(x_1) + f(x_2) + \dots + f(x_n)$

$\forall i = \overline{1,n}$ $x_i = 1 \Rightarrow \underline{f(n)} = n \cdot \underbrace{f(1)}_{a = const} = \underline{a \cdot n}$, $n \in \mathbb{N}$

$x_1 = x_2 = \dots = x_n = \frac{1}{n} \Rightarrow \underbrace{f(1)}_{a} = n f\left(\frac{1}{n}\right) \Rightarrow$

$\Rightarrow f\left(\frac{1}{n}\right) = a \cdot \frac{1}{n}$

$x_1 = x_2 = \dots = x_m = \frac{1}{n} \Rightarrow \underline{f\left(\frac{m}{n}\right)} = m f\left(\frac{1}{n}\right) = \underline{a \cdot \frac{m}{n}}$

$m, n \in \mathbb{N} \Rightarrow f\left(\frac{m}{n}\right) = a \cdot \frac{m}{n}$

$f\left(\frac{-m}{n}\right) = -f\left(\frac{m}{n}\right) = -a \cdot \frac{m}{n} = a\left(\frac{-m}{n}\right)$

$\forall x \in \mathbb{Q}$ $f(x) = a \cdot x$ $f(1) = a$

$x = \pi = 3{,}1415926\dots$
$r_1 = 3$
$r_2 = 3{,}1$
$r_3 = 3{,}14$
$\dots$

1) f - неперервна на $\mathbb{R}$

$x \in \mathbb{R} \setminus \mathbb{Q} \Rightarrow \exists \{r_n\} \subset \mathbb{Q} : r_n \to x$

непер. за Гейне $f(r_n) \to f(x)$

$\underline{f(x)} = \lim_n f(r_n) = \lim_n a \cdot r_n = \underline{ax}$

$a(x+y) = ax + ay$

G - базис Гамеля $x \in \mathbb{R}$

$x = n_1 g_1 + n_2 g_2 + \dots + n_k g_k$

$n_i \in \mathbb{Z}$, $g_i \in G$

$f(x) = n_1 \underline{f(g_1)} + n_2 \underline{f(g_2)} + \dots + n_k \underline{f(g_k)}$

2) f - монотонна $\forall x \in \mathbb{R} \setminus \mathbb{Q}$ $\forall q \in \mathbb{N}$

$\exists p \in \mathbb{Z}$: $\frac{p}{q} < x < \frac{p+1}{q}$

$f \nearrow$ $f\left(\frac{p}{q}\right) \le f(x) \le f\left(\frac{p+1}{q}\right)$

$f(0) = 0$

$\underbrace{f(1)}_{a} > f(0) > 0$

$a \cdot \frac{p}{q} \le f(x) \le a \cdot \frac{p+1}{q}$

$\frac{p}{q}$ $\frac{p+1}{q}$

$\frac{p}{q} \le \frac{f(x)}{a} \le \frac{p+1}{q}$

$\frac{f(x)}{a} = x \Rightarrow \underline{f(x) = ax}$

2. $f(x+y) = f(x) \cdot f(y)$

$x = y = \frac{t}{2} \Rightarrow f(t) = f^2\left(\frac{t}{2}\right) \ge 0$

$\exists y_0 \in \mathbb{R} : f(y_0) = 0 \Rightarrow f(x + y_0) = f(x) 0 =$

$\Rightarrow f(x + y_0) = 0$ $\forall x \in \mathbb{R} \Rightarrow \underline{f(x) \equiv 0}$

$\forall x$ $f(x) > 0$

$\underbrace{\ln f(x+y)}_{g(x+y)} = \underbrace{\ln f(x)}_{g(x)} + \underbrace{\ln f(y)}_{g(y)}$

$g(x) = ax = \ln f(x) \Rightarrow \underline{f(x) = e^{ax}} =$

$= \underline{b^x}$ $(b = e^a)$

3. $f(x \cdot y) = f(x) + f(y)$, $x, y \in (0; +\infty)$

$x = e^t$, $y = e^s$ $t, s \in \mathbb{R}$

$\underbrace{f(e^{t+s})}_{g(t+s)} = \underbrace{f(e^t)}_{g(t)} + \underbrace{f(e^s)}_{g(s)}$

$g(t) = \underline{f(e^t) = at}$

$\underline{f(x) = a \ln x}$ $f(x) \equiv 0$

4) $f(xy) = f(x) \cdot f(y)$, $x, y \in (0; +\infty)$

$x = e^t$, $y = e^s$

$f(e^t \cdot e^s) = f(e^t) \cdot f(e^s)$

$\underbrace{f(e^{t+s})}_{g(t+s)} = \underbrace{f(e^t)}_{g(t)} \cdot \underbrace{f(e^s)}_{g(s)}$

$g(t) = \underline{b^t = f(e^t)}$

$\underline{f(x)} = b^{\ln x} = b^{\frac{\log_b x}{\log_b e}} =$

$\log_b x$

$= b^{\ln b \log_b x} = b$ $= x^{\ln b} = \underline{x^{\alpha}}$

$f(x) \equiv 0$

$\underline{\text{Пр.1.}}$ $f(x+y) = \frac{f(x) \cdot f(y)}{f(x) + f(y)}$

$\exists y_0 :$ $f(y_0) = 0 \Rightarrow \underline{f(x+y_0)} = \frac{f(x) \cdot f(y_0)}{f(x)}$

$\underline{= 0}$ супер.

$\forall x \in X$ $f(x) \ne 0$

$\underbrace{\frac{1}{f(x+y)}}_{g(x+y)} = \frac{f(x) + f(y)}{f(x) \cdot f(y)} = \underbrace{\frac{1}{f(y)}}_{g(y)} + \underbrace{\frac{1}{f(x)}}_{g(x)}$

$g(x) = \frac{1}{f(x)} = ax \Rightarrow f(x) = \frac{1}{ax} =$

$= \frac{b}{x}$, $x \ne 0, b \ne 0$

Перевірка: $\frac{b}{x+y} \stackrel{?}{=} \frac{\frac{b}{x} \cdot \frac{b}{y}}{\frac{b}{x} + \frac{b}{y}} = \frac{\frac{b^2}{xy}}{\frac{b(y+x)}{xy}} =$

$f(x) = \frac{b}{x}$, $x, b \ne 0$

$y, x \in \mathbb{R}$

Пр. 2. $f(x+y) = f(x) + f(y) + f(x) f(y)$

$\underbrace{f(x+y) + 1}_{g(x+y)} = \underbrace{(f(x) + 1)}_{g(x)} \cdot \underbrace{(f(y) + 1)}_{g(y)}$

$g(x) \equiv 0$ $\vee$ $g(x) = b^x$

$f(x) \equiv -1$ $\vee$ $f(x) = b^x - 1$