1) $\left(x, f(x)\right)$
$\left(y, f(y)\right)$

$x \cdot f(y) + y \cdot f(x) = f(xy)$ $\quad x > 0$

$\underbrace{\frac{f(y)}{y}}_{g(y)} + \underbrace{\frac{f(x)}{x}}_{g(x)} = \underbrace{\frac{f(xy)}{xy}}_{g(xy)}$ $\quad \cancel{c}\cancel{x}\cdot\cancel{y}\ln y + \cancel{xy}\cancel{\ln x} = \cancel{c}\cdot\cancel{xy}\ln xy$

$g(x) = c \cdot \ln x \Rightarrow c \cdot x \cdot \ln x = f(x)$

2) $\forall x > 0 \quad f(f(x)) = x \cdot f(x)$

$f(x) \equiv \text{const } c \qquad C = Cx \Leftrightarrow C = 0$

$\boxed{f_1(x) \equiv 0}$

$\forall x > 0 \quad f(x) > 0$

$f(x) = y \Rightarrow f(y) = x \cdot y$

$f(f(y)) = f(xy)$

$f(f(y)) = y \cdot f(y) = f(x) \cdot f(y)$

$\Downarrow$

$f(xy) = f(x) \cdot f(y)$

$f(x) = x^{\alpha}$

$f(f(x)) = (x^{\alpha})^{\alpha} = x \cdot f(x) = x \cdot x^{\alpha}$

$x^{\alpha^2} = x^{\alpha+1}, \quad \forall x > 0$

$\alpha^2 = \alpha + 1 \Rightarrow \alpha^2 - \alpha - 1 = 0$

$D = 5 \qquad \alpha_{1,2} = \frac{1 \pm \sqrt{5}}{2}$

$f_2(x) = x^{\frac{1+\sqrt{5}}{2}}$

$f_3(x) = x^{\frac{1-\sqrt{5}}{2}}$

3) $\mathbb{R} \qquad \forall x, y \in \mathbb{R}$

$f(x) \cdot g(y) = x + y + 1$

$$\begin{aligned} x = y = 0 &\Rightarrow \\ x = 1, y = -1 &\Rightarrow \\ x = 0, y = -1 &\Rightarrow \end{aligned} \begin{cases} f(0) \cdot g(0) = 1 \\ f(1) \cdot g(-1) = 1 \\ f(0) \cdot g(-1) = 0 \end{cases}$$

$\nexists f, g$

4) $f(x+y) = f(x) \cdot e^{y}$

$x = 0 \Rightarrow f(y) = f(0) e^{y} = a \cdot e^{y}$

Перев: $a e^{x+y} = a e^{x} \cdot e^{y}$

$\boxed{f(x) = a \cdot e^{x}}$

5) $f(x+y) + f(x-y) - f(x) - x^3 - 6xy^2 \sqrt[6]{f(y)} = 0$

$y = 0 \Rightarrow f(x) - x^3 = 0$

$f(x) = x^3$

6) $f: \mathbb{R} \to \mathbb{R}$

$x \cdot f(y) + y f(x) = (x+y) f(x) \cdot f(y)$

$x = y = 1: \quad \cancel{2} f(1) = \cancel{2} f^2(1)$ $\qquad f^2(1) = f(1)$

$f(1) = 0 \qquad f(1) = 1$

I. $f(1) = 0$ : $y = 1 \Rightarrow x \cdot \underbrace{f(1)}_{0} + 1 \cdot f(x) = (x+1) \cdot f(x) \underbrace{f(1)}_{0}$

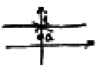
$f_1(x) \equiv 0$

II. $f(1) = 1$ : $y = 1 \Rightarrow x + f(x) = (x+1) f(x)$

$x + \cancel{f(x)} = x f(x) + \cancel{f(x)}$

$x \neq 0 \qquad 1 = f(x)$

$f_2(x) = \begin{cases} 1, & x \neq 0 \\ a, & x = 0 \end{cases}$



Перевірка : $x \neq 0 \qquad x + y = x + y$

$x = 0 \qquad 0 \cdot a + 0 \cdot a = 0 \cdot a^2$

метод підстановки

I. $\boxed{f(\varphi(x)) = F(x)}$ $\qquad \varphi, F$ – відомі

маю

$\boxed{y = \varphi(x)}$ - заміна

$\exists\ x = \varphi^{-1}(y)$

$\boxed{f(y) = F(\varphi^{-1}(y))}$

$f\left(\frac{x}{x+1}\right) = x^2$

$\varphi(x) = \frac{x}{x+1} \qquad D(\varphi) = (-\infty; -1) \cup (-1; +\infty)$

$\frac{x}{x+1} = y \Rightarrow x = yx + y$

$x(1-y) = y \Rightarrow x = \overbrace{\frac{y}{1-y}}^{\varphi^{-1}(y)}$

$D(\varphi^{-1}) = (-\infty; 1) \cup (1; +\infty)$

$f(y) = \left(\frac{y}{1-y}\right)^2 \Rightarrow f(x) = \left(\frac{x}{1-x}\right)^2$

на $(-\infty; 1) \cup (1; +\infty)$

Перевірка : $f\left(\frac{x}{x+1}\right) = \left(\frac{\frac{x}{x+1}}{1 - \frac{x}{x+1}}\right)^2 =$

$= \left(\frac{\frac{x}{\cancel{x+1}}}{\frac{\cancel{x+1-x}}{\cancel{x+1}}}\right)^2 = x^2$

II. 1) $a(x) f(\varphi(x)) + b(x) \cdot f(x) = F(x)$

$a(x), b(x), F(x), \varphi(x)$ – відомі

$\exists$ обер. ф-ція $\quad x = \varphi^{-1}(y)$

$\varphi^{-1}(x) = \varphi(x) \qquad \frac{1}{x}$

$\cancel{\varphi(\varphi(x)) = x}$

$x \to \varphi(x)$ в 1) :

2) $a(\varphi(x)) f(x) + b(\varphi(x)) \cdot f(\varphi(x)) = F(\varphi(x))$

Пр. $\quad x \neq 0$

$f(x) - 2f\left(\frac{1}{x}\right) = 2^x$

$y = \varphi(x) = \frac{1}{x} \Rightarrow x = \frac{1}{y} = \varphi^{-1}(y)$

$\varphi^{-1}(x) = \frac{1}{x}$

$x \to \frac{1}{x} \quad \begin{cases} f(x) - 2f\left(\frac{1}{x}\right) = 2^x \\ f\left(\frac{1}{x}\right) - 2f(x) = 2^{\frac{1}{x}} \quad \Big| \times 2 \end{cases} \oplus$

$f(x) - 2f\left(\frac{1}{x}\right) = 2^x$

$-3f(x) = 2 \cdot 2^{\frac{1}{x}} + 2^x$

$f(x) = -\frac{2^x + 2^{1+\frac{1}{x}}}{3}$ $\qquad$ Перевірка