$$f(\varphi(x)) = F(x)$$

$$- \quad a(x)\cdot f(\varphi(x)) + b(x)\cdot f(x) = F(x)$$

$$- \quad a(\varphi(\varphi(x))) f(\varphi(\varphi(\varphi(x)))) + b(\varphi(\varphi(x))) f(\varphi(\varphi(x))) =$$
$$= F(\varphi(\varphi(x)))$$

Пр. 1 $\quad f(x) + f\left(\frac{1}{1-x}\right) = x$

$$\varphi(x) = \frac{1}{1-x}$$

$$\varphi(\varphi(x)) = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{\cancel{1}-x-\cancel{1}} = \frac{x-1}{x}$$

$$\varphi(\varphi(\varphi(x))) = \frac{1}{1-\frac{x-1}{x}} = \frac{x}{\cancel{x}-\cancel{x}+1} = x$$

$$x \to \frac{1}{1-x}$$

$$f\left(\frac{1}{1-x}\right) + f\left(\frac{x-1}{x}\right) = \frac{1}{1-x}$$

$$x \to \frac{1}{1-x}$$

$$f\left(\frac{x-1}{x}\right) + f(x) = \frac{x-1}{x}$$

$$\begin{cases} f(x) + f\left(\frac{1}{1-x}\right) = x \\ f\left(\frac{1}{1-x}\right) + f\left(\frac{x-1}{x}\right) = \frac{1}{1-x} \\ f\left(\frac{x-1}{x}\right) + f(x) = \frac{x-1}{x} \end{cases} \qquad (1)-(2)+(3)$$

$$2f(x) = x - \frac{1}{1-x} + \frac{x-1}{x}$$

$$f(x) = \frac{1}{2}\left(x - \frac{1}{1-x} + \frac{x-1}{x}\right)$$

$$\frac{1}{2}\left(x - \cancel{\frac{1}{1-x}} + \cancel{\frac{x-1}{x}}\right) + \frac{1}{2}\left(\cancel{\frac{1}{1-x}} - \cancel{\frac{x-1}{x}} + x\right) =$$
$$= \frac{1}{2}(x+x) = x$$

...

$$\underbrace{\varphi(\varphi \ldots \varphi}_{k}(x))) = x \qquad x \to \varphi(x)$$

Пр. $\quad f(x) + f\left(\frac{a^2}{a-x}\right) = x, \quad a \neq 0$

$$\varphi(x) = \frac{a^2}{a-x}$$

$$\varphi(\varphi(x)) = \frac{a^2}{a-\frac{a^2}{a-x}} = \frac{a^2(a-x)}{a(a-x-a)} =$$
$$= \frac{a(a-x)}{-x} = \frac{a(x-a)}{x}$$

$$\varphi(\varphi(\varphi(x))) = \frac{a^2}{a-\frac{a(x-a)}{x}} = \frac{a^2 x}{ax-ax+a^2} = x$$

$$\begin{cases} f(x) + f\left(\frac{a^2}{a-x}\right) = x \\ f\left(\frac{a^2}{a-x}\right) + f\left(\frac{a(x-a)}{x}\right) = \frac{a^2}{a-x} \\ f\left(\frac{a(x-a)}{x}\right) + f(x) = \frac{a(x-a)}{x} \end{cases}$$

$$f(x) = \frac{1}{2}\left(x - \frac{a^2}{a-x} + \frac{a(x-a)}{x}\right)$$

Пр. 3 $\quad 2f(x+y) + f(x-y) = f(x)(2e^y + e^{-y})$

$x = 0 \qquad 2f(y) + f(-y) = \underbrace{f(0)}_{a}(2e^y + e^{-y})$

$2x = y \qquad 2f(3x) + f(-x) = f(x)(2e^{2x} + e^{-2x})$

$y = -2x \qquad 2f(-x) + f(3x) = f(x)(2e^{-2x} + e^{2x})$

$$\begin{cases} ① \; 2f(x) + f(-x) = a(2e^x + e^{-x}) \\ ② \; 2f(3x) + f(-x) = f(x)(2e^{2x} + e^{-2x}) \\ ③ \; 2f(-x) + f(3x) = f(x)(2e^{-2x} + e^{2x}) \end{cases}$$

$2x \qquad ② - 2\cdot③$

$$-3f(-x) = -2f(x)(2e^{-2x} + e^{2x}) +$$
$$+ f(x)(2e^{2x} + e^{-2x}) =$$
$$= f(x)\left(-4e^{-2x} - \cancel{2e^{2x}} + \cancel{2e^{2x}} + e^{-2x}\right) =$$
$$= -3e^{-2x}\cdot f(x)$$

$$f(-x) = e^{-2x}\cdot f(x) \quad \to ①$$

$$2f(x) + e^{-2x}\cdot f(x) = a(2e^x + e^{-x})$$

$$f(x) = \frac{a(2e^x + e^{-x})}{2 + e^{-2x}} =$$

$$= \frac{a\left(2e^x + \frac{1}{e^x}\right)}{2 + \frac{1}{e^{2x}}} = a\,\frac{\frac{\cancel{2e^{2x}+1}}{e^x}}{\frac{\cancel{2e^{2x}+1}}{e^{2x}}} =$$

$$= ae^x$$

$$2\cancel{a}e^{x+y} + a^{x-y} = \cancel{a}e^x(2e^y + e^{-y})$$

Пр. 4. $\quad 2f(x) + f(1-x) = x^2 \qquad ①$

$$\varphi(x) = 1 - x$$

$$\varphi(\varphi(x)) = 1 - (1-x) = x$$

$$x \to 1 - x$$

$$2f(1-x) + f(x) = (1-x)^2 \qquad ②$$

① $\times(-2)$ + ②:

$$-3f(x) = -2x^2 + (1-x)^2 = -2x^2 + 1 - 2x + x^2$$
$$= -x^2 - 2x + 1$$

$$f(x) = \frac{x^2 + 2x - 1}{3}$$

Перевірка:

$$\frac{2x^2 + 4x - 2}{3} + \frac{(1-x)^2 + 2(1-x) - 1}{3} =$$

$$= \frac{2x^2 + \cancel{4x} - \cancel{2} + \cancel{1} - \cancel{2x} + x^2 + \cancel{2} - \cancel{2x} - \cancel{1}}{3} = \frac{3x^2}{3} = x^2$$