Пр.1 $f(f(x)) = e^{-x}$, $x \in \mathbb{R}$

$f \searrow (\nearrow) \Rightarrow f(f) \nearrow$

ін'єктивність? $f(x_1) = f(x_2)$

$e^{-x_1} = f(f(x_1)) = f(f(x_2)) = e^{-x_2}$

$x_1 = x_2$

$\left.\begin{array}{l} f - \text{ін'єкт.} \\ f - \text{непер.} \end{array}\right\} \Rightarrow f$ – монот. $\Rightarrow f(f) \nearrow \neq e^{-x}$ ↯

Пр.2 $f(x+1)\cdot f(x) + f(x+1) + 1 = 0 \quad \forall x \in \mathbb{R}$

$\exists x_0 \in \mathbb{R} \quad f(x_0) = 0$

$x = x_0 - 1$

$\underbrace{f(x_0)}_{0}\cdot f(x_0-1) + \underbrace{f(x_0)}_{0} + 1 = 0$

протиріччя

1) $\forall x \quad f(x) > 0$ ціла частина > 1 x_0

2) $\forall x \quad f(x) < 0$

$f(x+1)\cdot f(x) + f(x+1) = -1$

$f(x+1)\,(\underset{-}{f(x)} + \underset{+}{1}) = -1$

$f(x) + 1 > 0 \Rightarrow f(x) > -1$

$-1 < f(x) < 0$

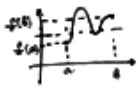
$-1 < f(x+1) < 0$

$0 < f(x+1) + 1 < 1$

$\underset{+}{f(x+1)}\,\underset{+}{f(x)} + \underset{+}{f(x+1)+1} \neq 0$

Пр.3 $f : [1;2] \to [1;2]$

$f(1) = 2$ та $f(f(x))\cdot f(x) = 2 \quad \forall x \in [1;2]$



$f \in C[1;2]$

$x = 1$

$f(\underbrace{f(1)}_{2})\cdot\underbrace{f(1)}_{2} = 2 \Rightarrow f(2) = 1$

$\forall y \in [\underbrace{f(2)}_{1}; \underbrace{f(1)}_{2}] \quad \exists x \in [1;2] : f(x) = y$

$f(y)\cdot y = 2 \Rightarrow f(y) = \frac{2}{y}$

$\boxed{f(x) = \frac{2}{x}}$

перевірка: $\frac{2}{\frac{2}{x}} \cdot \frac{2}{x} = \cancel{x}\cdot\frac{2}{\cancel{x}} = 2$

f – непер. в т. x_0

$\exists \lim\limits_{x \to x_0} f(x) = f(x_0)$

Тобто: $\forall x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

Пр.4 $x = 0; 1; -1$ – т. непер.

$f : \mathbb{R} \to \mathbb{R}$

$f(x^3 + y) = f(x^3) + f(y^3) \quad , x, y \in \mathbb{R}$

$x = y = 0 \quad f(0) = 2f(0) \Rightarrow f(0) = 0$

$y = 0 \quad f(x^3) = f(x) \quad x \in \mathbb{R}$

$x = \sqrt[3]{t}$: $f(t) = f(\sqrt[3]{t}) = f(\underbrace{\sqrt[3]{\sqrt[3]{t}}}_{f(\sqrt[9]{t})}) = f(\sqrt[9]{t}) = \ldots =$

$= f(\sqrt[3^n]{t}) \quad \forall n \in N$

мат. індукція : $n = 1$: $f(t) = f(\sqrt[3]{t})$ істина

припустимо : $f(t) = f(\sqrt[3^n]{t})$

доведемо : $f(t) \overset{?}{=} f(\sqrt[3^{n+1}]{t})$ ✓

$f(t) = f(\sqrt[3^n]{t})$

$t \to \sqrt[3]{t} \quad f(\sqrt[3]{t}) = f(\sqrt[3]{\sqrt[3^n]{t}}) = f(\sqrt[3^{n+1}]{t})$ ✓

Отже, $f(t) = f(\sqrt[3^n]{t}) \quad \forall n$

$\lim\limits_n f(t) = \lim\limits_n f(\sqrt[3^n]{t}) =$

$f(t) \qquad = f(\lim\limits_n \sqrt[3^n]{t}) =$

$a > 0$

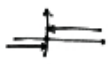
$\lim\limits_n \sqrt[n]{a} = 1$

$t > 0 : \lim\limits_n \sqrt[3^n]{t} = 1$

$t < 0 : \lim\limits_n \sqrt[3^n]{t} = -1$

$t > 0 \quad f(t) = f(1)$

$t < 0 \quad f(t) = f(-1)$



$f(0) = \underbrace{f(0+0)}_{f(1)} = \underbrace{f(0-0)}_{f(-1)} = 0$

$\boxed{f(x) \equiv 0}$

Пр.5 $2f(2x) - f(\frac{x}{2}) = 2(2x+1)$, $x \in \mathbb{R}$

$2x = t \quad 2f(t) - f(\frac{t}{4}) = 2(t+1)$

$f(t) = \frac{1}{2}f(\frac{t}{4}) + t + 1$

$t \to t/4 \quad f(\frac{t}{4}) = \frac{1}{2}f(\frac{t}{16}) + \frac{t}{4} + 1$

$f(t) = \frac{1}{2}\left(\frac{1}{2}f(\frac{t}{16}) + \frac{t}{4} + 1\right) =$

$= \frac{1}{2^2}f(\frac{t}{16}) + \frac{1}{2}\cdot\frac{t}{4} + \frac{1}{2} + t + 1 = \frac{1}{2^2}f(\frac{t}{16}) + t(1 + \frac{1}{2}) + 1 + \frac{1}{2}$

$f(\frac{t}{16}) = \left(\frac{1}{2}f(\frac{t}{16}) + \frac{t}{16} + 1\right) =$

$= \frac{1}{2^2}\left(\frac{1}{2}f(\frac{t}{16}) + \frac{t}{16} + 1\right) + (1 + \frac{1}{2})t + (1 + \frac{1}{2}) =$

$= \frac{1}{2^3}f(\frac{t}{16}) + \left(1 + \frac{1}{2^4} + \frac{1}{2^2\cdot 16}\right)t + (1 + \frac{1}{2} + \frac{1}{2^2}) =$

$= \ldots = \frac{1}{2^n}f(\frac{t}{4^n}) + \left(1 + \frac{1}{8} + \frac{1}{8^2} + \ldots + \frac{1}{8^{n-1}}\right)t + (1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}) =$

мат. індукція

$1 + q + \ldots + q^{n-1} = \frac{1-q^n}{1-q}$

$= \frac{1}{2^n}f(\frac{t}{4^n}) + \frac{1 - \frac{1}{8^n}}{\frac{7}{8}}t + \frac{1 - \frac{1}{2^n}}{\frac{1}{2}} =$

$= \frac{1}{2^n}f(\frac{t}{4^n}) + \frac{8}{7}(1 - \frac{1}{8^n})t + 2(1 - \frac{1}{2^n}) = f(t)$

$n = 1 : \quad \frac{1}{2}f(\frac{t}{4}) + t + 1 = f(t)$

припуст. $f(t) = \frac{1}{2^n}f(\frac{t}{4^n}) + \frac{8}{7}(1 - \frac{1}{8^n})t + 2(1 - \frac{1}{2^n})$

довести : $f(t) \overset{?}{=} \frac{1}{2^{n+1}}f(\frac{t}{4^{n+1}}) + \frac{8}{7}(1 - \frac{1}{8^{n+1}})t + 2(1 - \frac{1}{2^{n+1}})$

$f(t) = \frac{1}{2}f(\frac{t}{4}) + t + 1$ ⊜

$f(\frac{t}{4}) = \frac{1}{2^n}f(\frac{t}{4^{n+1}}) + \frac{8}{7}(1 - \frac{1}{8^n})\cdot\frac{t}{4} + 2(1 - \frac{1}{2^n})$

⊜ $\frac{1}{2}\left(\frac{1}{2^n}f(\frac{t}{4^{n+1}}) + \frac{8}{7}(1 - \frac{1}{8^n})\cdot\frac{t}{4} + 2(1 - \frac{1}{2^n})\right) + t + 1 =$

$= \frac{1}{2^{n+1}}f(\frac{t}{4^{n+1}}) + \frac{1}{2}\cdot\frac{8}{7}(1 - \frac{1}{8^n})\cdot\frac{t}{4} + t + \frac{1}{2}\cdot 2(1 - \frac{1}{2^n}) + 1 =$

$= \frac{1}{2^{n+1}}f(\frac{t}{4^{n+1}}) + \frac{8}{7}(1 - \frac{1}{8^{n+1}})t + 2(1 - \frac{1}{2^{n+1}})$

$\frac{8}{7}(1 - \frac{1}{8^n}) + 1 = \frac{1}{7}(1 - \frac{1}{8^n} + 7) = \frac{1}{7}(8 - \frac{1}{8^n}) =$

$= \frac{8}{7}(1 - \frac{1}{8^{n+1}})$

$f(t) = \underbrace{\frac{1}{2^{n+1}}f(\frac{t}{4^{n+1}})}_{\to 0} + \frac{8}{7}(1 - \underbrace{\frac{1}{8^{n+1}}}_{\to 0})t + 2(1 - \underbrace{\frac{1}{2^{n+1}}}_{\to 0})$

$n \to \infty \quad f(t) = \frac{8}{7}t + 2$

$f(x) = \frac{8}{7}x + 2$

Перевірка: $2f(2x) - f(\frac{x}{2}) =$

$= 2(\frac{8}{7}\cdot 2x + 2) - \frac{8}{7}\cdot\frac{x}{2} - 2 = \frac{32}{7}x + 4 - \frac{4}{7}x - 2 =$

$= 4x + 2 = 2(2x+1)$