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Perron-Frobenius Theorem and Invariant Sets in Linear Systems Dynamics

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Abstract - **The paper explores the connections between the Perron-Frobenius (PF) theory and the flow-invariant sets with respect to the dynamics of linear systems. Our analysis includes both discrete- and continuous-time systems, and the results are separately formulated for linear dynamics generated by the following types of matrices: (i) (essentially) nonnegative and irreducible or (essentially) positive, (ii) (essentially) nonnegative and reducible. For both cases we show how the PF eigenvalue and right and left PF eigenvectors are related to invariant sets defined for any Hölder *p*-norm ($1 \le p \le \infty$).**

I. INTRODUCTION

Some of the contributions brought by the mathematicians Oskar Perron (1880-1975) and Ferdinand Georg Frobenius (1849-1917) to the progress of linear algebra at the beginning of the previous century are nowadays known as the Perron-Frobenius (PF) theory (Theorems 8.2.2 and 8.4.4 in [1] – see the Appendix). This theory represents an important instrument for exploring the dynamics of positive systems, being used either for developing theoretical approaches [2] – [9], or for dealing with specific applications that arise in different areas of science and engineering [10], [11].

The paper [4] presents nice comments on the key role played by the PF theory in the qualitative analysis, as "providing information on the long term behavior of an homogeneous positive irreducible system", in the sense that any trajectory evolves with a speed given by the PF eigenvalue and approaches the PF eigenvector (which belongs to the first orthant of the state space).

If we continue to use this intuitive style for exposition, then we may say that the current paper brings complementary information, by showing the utility of the PF Theorem in the characterization of both short and long term behavior. We prove the existence of different families of time-dependent sets, which are invariant with respect to the system motion, i.e. any trajectory initialized inside such a set will never leave it.

In a rigorous formulation, our objective is to connect the mathematical description of these families of invariant sets with the PF eigenvalue and the right and left PF eigenvectors. For asymptotically stable systems, the different families of invariant sets are contractive and their existence is mutually related to the existence of different types of Lyapunov functions. Although a great deal of research has been invested in the study of invariant sets of linear systems (as reflected by the survey work [12] and the papers cited therein), the link with the PF eigenstructure remained almost unexplored. A brief analysis appears in the more recent work [13] and it will be further commented in the main sections of our approach.

In our paper, we consider both *discrete-time* (abbreviated DT) and *continuous-time* (abbreviated CT) positive linear systems. The DT case is defined by:

$$x(t+1) = Ax(t),\ x(t_0) = x_0,\ t_0, t \in \mathbf{Z}_+,\ t \ge t_0, \qquad \text{(1-DT)}$$

and the CT case is defined by:

$$\dot{x}(t) = Ax(t),\ x(t_0) = x_0,\ t_0, t \in \mathbf{R}_+,\ t \ge t_0. \qquad \text{(1-CT)}$$

where $A \in \mathbf{R}^{n \times n}$.

We develop connections between the PF Theorem and the dynamics of the systems (1-DT) and (1-CT) under the following assumptions on the matrix generating the dynamics. In section II, matrix A is considered positive or nonnegative and irreducible for system (1-DT) and essentially positive or essentially nonnegative and irreducible for system (1-CT). In section III, matrix A is considered nonnegative and reducible for system (1-DT) and essentially nonnegative and irreducible for system (1-CT).

Throughout the paper we use the following notations. For a vector $x \in \mathbf{R}^n$: $\| x \|$ is an arbitrary vector norm; $\| x \|_p$ is the Hölder vector p-norm, $1 \le p \le \infty$; $| x |$ stands for the nonnegative vector defined by taking the absolute values of the elements of x. If $x, y \in \mathbf{R}^n$, then "$x \le y$", "$x < y$" mean componentwise inequalities. For a matrix $M \in \mathbf{R}^{n \times n}$: $\| M \|_p$ is the matrix norm induced by the vector norm $\| \bullet \|_p$; $\mu_{\| \|_p}(M) = \lim_{h \downarrow 0}(\|I + hM\|_p - 1)/h$ is a matrix measure ([14], pp. 29), based on the matrix norm $\| \bullet \|_p$; $| M |$ stands for the matrix defined by taking the absolute values of the entries of M. If $M, P \in \mathbf{R}^{n \times n}$, then "$M \le P$", "$M < P$" mean componentwise inequalities. The spectrum of M is $\sigma(M) = \{ z \in \mathbf{C} \mid \det(zI - M) = 0 \}$, and $\lambda_i(M) \in \sigma(M)$, $i = 1, \ldots, n$, denote its eigenvalues.

II. Dynamics Defined by (Essentially) Positive or (Essentially) Nonnegative and Irreducible Matrices

In this section we consider that the matrix A of system (1-DT) is either *positive* (all the entries are greater than 0) or *nonnegative* (all the entries are greater or equal to 0) and *irreducible* (the oriented graph associated with A is strongly connected - other equivalent characterizations are given by Theorem 6.2.24 in [1]). Similarly, the matrix A of system (1-CT) is either *essentially positive* (all the off-diagonal entries are greater than 0) or *essentially nonnegative* (all the off-diagonal entries are greater or equal to 0) and *irreducible*. First, we introduce the concept of PF p-eigenpattern associated with the above types of matrices and then, we reveal the connections between this new concept and the dynamics of systems (1-DT) and (1-CT).

A. The Perron Frobenius p-Eigenpattern

In the DT case, denote by $\lambda_{PF}(A)$ the PF eigenvalue of matrix A (i.e. it satisfies the condition $|\lambda_i(A)| \le \lambda_{PF}(A)$, $i=1,...,n$), and by $r=[r_1 \dots r_n]^T>0$ and $l=[l_1 \cdots l_n]^T>0$ the right and left PF eigenvectors (fulfilling the equalities $Ar=\lambda_{PF}(A)r$, $A^T l=\lambda_{PF}(A)l$). For the moment, in order to ensure maximum of clarity, we refer to r and l as unique vectors, by choosing their elements such that $\sum_{i=1}^n r_i=1$, $\sum_{i=1}^n l_i=1$. However, we will show that the uniqueness can be omitted and the PF eigenvectors can be understood in their general sense of directions (not affected by positive constant multiplications).

In the CT case, we can still use the terminology of PF eigenvalue and eigenvectors, as motivated below. Let us pick a real s such that $sI+A$ is positive or nonnegative and denote the eigenvalues of A and $sI+A$ such that $\lambda_i(sI+A)=s+\lambda_i(A)$, $i=1,\dots,n$; define $\lambda_{PF}(A)$ so that $s+\lambda_{PF}(A)=\lambda_{PF}(sI+A) \ge |\lambda_i(sI+A)| \ge s+\mathrm{Re}\{\lambda_i(A)\}$, for all $i=1,\dots,n$. Thus, A has a real eigenvalue, denoted by $\lambda_{PF}(A)$, such that $\mathrm{Re}\{\lambda_i(A)\} \le \lambda_{PF}(A)$, $i=1,...,n$. Moreover, once A is irreducible, $sI+A$ is also irreducible and its PF eigenvectors are eigenvectors of A. Consequently, there exist $r=[r_1 \dots r_n]^T>0$ and $l=[l_1 \cdots l_n]^T>0$, with $\sum_{i=1}^n r_i=1$, $\sum_{i=1}^n l_i=1$, such that $Ar=\lambda_{PF}(A)r$ and, respectively, $A^T l=\lambda_{PF}(A)l$. Relying on the above discussion, the meanings of the PF eigenvalue and eigenvectors in the DT and CT cases agree.

This fact allows the development of a unified approach in presenting our new results. Thus, regardless of the DT or CT nature of a positive linear system, for each $1 \le p \le \infty$ we can define a positive definite diagonal matrix $D_{PFp}(A)$, as follows. Consider $1 \le q \le \infty$ such that $1/p+1/q=1$, where the particular cases $p=1$ and $p=\infty$ mean $1/p=1$, $1/q=0$, and $1/p=0$, $1/q=1$, respectively. By using the left and right PF eigenvectors, construct the matrix:

$$D_{PFp}(A)=\mathrm{diag}\left\{\frac{(l_1)^{1/p}}{(r_1)^{1/q}},\dots,\frac{(l_n)^{1/p}}{(r_n)^{1/q}}\right\}. \qquad (2)$$

Definition 1. Let A be an (essentially) positive matrix or an (essentially) nonnegative and irreducible matrix. The pair $\lambda_{PF}(A)$, $D_{PFp}(A)$ defines the PF p-eigenpattern of A.

B. Properties of Discrete-Time Systems

In the current subsection, we study the connections between the PF p-eigenpatterns of A and the properties of the dynamical system (1-DT).

Theorem 1. Let $1 \le p \le \infty$. Let matrix A be positive or nonnegative and irreducible. The following three statements are true. Moreover, these statements are equivalent.

(a) $\| D_{PFp}(A)\, A\, D_{FPp}^{-1}(A) \|_p = \lambda_{PF}(A)$ (3-DT)

(b) Along any trajectory $x(t)=x(t;\, t_0, x_0)$ of system (1-DT), the motion fulfills the condition:

$$\| D_{PFp}(A)x(t+1) \|_p \le \lambda_{PF}(A) \| D_{PFp}(A)\, x(t) \|_p,$$
$$\forall t \in \mathbf{Z}_+,\ t \ge t_0. \qquad (4\text{-DT})$$

(c) The time-dependent sets

$$X_{PFp}^{\varepsilon}(t;t_0)=\left\{x \in \mathbf{R}^n \,\middle|\, \| D_{PFp}(A)\, x \|_p \le \varepsilon \left(\lambda_{PF}(A)\right)^{(t-t_0)}\right\},$$
$$t, t_0 \in \mathbf{Z}_+,\ t \ge t_0,\ \varepsilon>0, \qquad (5\text{-DT})$$

are (positively) invariant with respect to (w.r.t.) the state-space trajectories of the DT system (1-DT).

Proof: From Theorem 1 in [15], we have that for any $1 \le p \le \infty$ the matrix $D_{PFp}(A)$ satisfies the equality (3-DT). Now we have to show that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

(a) $\Rightarrow$ (b): For any trajectory $x(t)=x(t;t_0,x_0)$ of system (1-DT) and for any $t \in \mathbf{Z}_+$, $t \ge t_0$, we have $\|D_{PFp}(A)x(t{+}1)\|_p \le \|D_{PFp}(A)\,A\,D_{PFp}^{-1}(A)\|_p \|D_{PFp}(A)x(t)\|_p = \lambda_{PF}(A)\|D_{PFp}(A)x(t)\|_p$.

(b) $\Rightarrow$ (a): For any trajectory $x(t)=x(t;t_0,x_0)$ of system (1-DT) and for any $t \in \mathbf{Z}_+$, $t \ge t_0$, we can write $\|D_{PFp}(A)\ A\ D_{FPp}^{-1}(A)\|_p=$

$$\sup_{x(t)\ne 0} \frac{\|\left(D_{PFp}(A)\ A\ D_{FPp}^{-1}(A)\right)\left(D_{FPp}(A)x(t)\right)\|_p}{\| D_{FPp}(A)x(t) \|_p} =$$

$\sup_{x(t)\ne 0} \dfrac{\| D_{PFp}(A)\ x(t+1) \|_p}{\| D_{PFp}x(t) \|_p} \le \lambda_{PF}(A)$. On the other hand, for any positive definite diagonal matrix D, $\lambda_{PF}(A)=\lambda_{PF}(DAD^{-1}) \le \| DAD^{-1} \|_p$. The two inequalities yield (3-DT).

(b) $\Rightarrow$ (c): We construct a proof by contradiction, assuming that the time-dependent sets $X_{PFp}^{\varepsilon}(t;t_0)$ defined by (5-DT) are not positively invariant w.r.t. the state-space trajectories of system (1-DT).

This means there exist a trajectory $\hat{x}(t)=\hat{x}(t;\,t_0,x_0)$ of (1-DT), a constant $\varepsilon>0$ and a moment $t^*\geq t_0$, such that $\| D_{PFp}(A)\hat{x}(t^*)\|_p \leq \varepsilon\,(\lambda_{PF}(A))^{(t^*-t_0)}$, whereas $\| D_{PFp}(A)\hat{x}(t^*{+}1)\|_p > \varepsilon\,(\lambda_{PF}(A))^{(t^*+1-t_0)}$. Hence, we get $\lambda_{PF}(A)\| D_{PFp}(A)\hat{x}(t^*)\|_p < \| D_{PFp}(A)\hat{x}(t^*+1)\|_p$, which contradicts (4-DT)

(c) $\Rightarrow$ (b): Let $x(t)$ be a solution to (1-DT) and let $t\in\mathbf{Z}_+$, $t>t_0$. Set $\varepsilon{=}\| D_{PFp}x(t)\|_p$. If $\varepsilon=0$, then $x(t)=0$, and hence $x(t+1)=0$. If $\varepsilon>0$, then by (5-DT), $\| D_{PFp}(A)\ x(t+1;t,x(t))\|_p \leq \| D_{PFp}(A)\ x(t)\|_p\ \lambda_{PF}(A)$, for all $t\in\mathbf{Z}_+$. □

Corollary 1. Let $1\leq p\leq\infty$. Let matrix A be positive or nonnegative and irreducible. The following three statements are equivalent.
(a) System (1-DT) is stable.
(b) The function defined by

$$V_{PFp}(x)=\| D_{PFp}(A)\ x\|_p\,, \qquad (6\text{-DT})$$

is a strong Lyapunov function for system (1-DT), with the decreasing rate $\lambda_{PF}(A)$.
(c) The invariant sets defined by (5-DT) are contractive, with the coefficient $\lambda_{PF}(A)$.

Proof: It results from Theorem 1, by taking into account the stability of the matrix A. □

Remark 1. Statement (b) of Corollary 1 shows that the PF eigenstructure of matrix A allows the construction of diagonal Lyapunov functions by using any Hőlder norm. This represents a nice generalization of the classical diagonal-type Lyapunov function defined by $V(x)=x^T\boldsymbol{\Delta}x$, with $\boldsymbol{\Delta}$ positive definite and diagonal (e.g. [16]). Indeed, the Lyapunov function $V_{PF2}(x)=\| D_{PF2}(A)\ x\|_2$, defined by (6-DT) for $p=2$, is equivalent to $V(x)=x^T\boldsymbol{\Delta}x$, with $\boldsymbol{\Delta}=D_{PF2}^2$, since $V(x)=V_{PF2}^2(x)$ and, along each trajectory of system (1-DT) we have $V(x(t+1))-V(x(t))=$ $[V_{PF2}(x(t{+}1))+V_{PF2}(x(t))][V_{PF2}(x(t{+}1))-V_{PF2}(x(t))]$, i.e. $\mathrm{sign}[V(x(t{+}1))-V(x(t))]=\mathrm{sign}[V_{PF2}(x(t{+}1))-V_{PF2}(x(t))]$, $\forall t\in\mathbf{Z}_+$. Moreover, matrix $\boldsymbol{\Delta}=D_{PF2}^2(A)$ is a solution to the Stein inequality $A^T\boldsymbol{\Delta}A-\boldsymbol{\Delta}\prec 0$, where "$\prec 0$" means "negative definite". Our generalization confirms that the diagonal Lyapunov functions and the contractive-type invariant sets are mutually related for any Hölder norm. □

C. *Properties of Continuous-Time Systems*

In the current subsection, we study the connections between the PF *p*-eigenpatterns of *A* and the properties of the dynamical system (1-CT).

Theorem 2. Let $1\leq p\leq\infty$. Let matrix A be essentially nonnegative and irreducible or essentially positive The following three statements are true. Moreover, these statements are equivalent.

(a) $\mu_{\|\,\|_p}(D_{PFp}(A)\,A\ D_{FPp}^{-1}(A))=\lambda_{PF}(A)\,. \qquad (3\text{-CT})$

(b) Along any trajectory $x(t)=x(t;t_0,x_0)$ of system (1-CT), the motion fulfills the condition:

$$\mathrm{D}_t^+\,\| D_{PFp}(A)\,x(t)\|_p \leq \lambda_{PF}(A)\,\| D_{PFp}(A)\,x(t)\|_p\,,\quad \forall t\in\mathbf{R}_+,\ t\geq t_0\,, \qquad (4\text{-CT})$$

where $D_t^+\,y(t)=\lim_{h\downarrow 0}[y(t+h)-y(t)]/h$ denotes the Dini right derivative.
(c) The time-dependent sets

$$X_{PFp}^{\varepsilon}(t;t_0)=\left\{x\in\mathbf{R}^n \,\middle|\, \| D_{PFp}(A)\ x\|_p\leq \varepsilon e^{\lambda_{PF}(A)(t-t_0)}\right\},\quad t,t_0\in\mathbf{Z}_+,\ t\geq t_0\,,\ \varepsilon>0\,, \qquad (5\text{-CT})$$

are (positively) invariant w.r.t. the state-space trajectories of the DT system (1-CT).

Proof: Let us pick an $s>0$ such that $sI+A$ is positive or nonnegative. Thus, by using Theorem 1, we get, for any $1\leq p\leq\infty$, $s+\lambda_{PF}(A)=\lambda_{PF}(sI+A)=$ $\| D_{PFp}(sI+A)\,(sI+A)\ D_{PFp}^{-1}(sI+A)\|_p=$ $\| D_{PFp}(A)\ (sI{+}A)\ D_{PFp}^{-1}(A)\|_p{=}\| sI{+}D_{PFp}(A)\ A\ D_{PFp}^{-1}(A)\|_p$, which yields $\lambda_{PF}(A)=\| sI+D_{PFp}(A)\ A\ D_{PFp}^{-1}(A)\|_p-s$. By taking $s=1/h$, for $h\downarrow 0$, we obtain (3-CT). Now we have to show that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

(a) $\Rightarrow$ (b): For any trajectory $x(t)=x(t;t_0,x_0)$ of system (1-CT) and for any $t\in\mathbf{R}_+$, $t\geq t_0$, we can write

$$\begin{aligned}
&\mathrm{D}_t^+\,\| D_{PFp}(A)\,x(t)\|_p=\\
&\lim_{h\downarrow 0}(\| D_{PFp}(A)\,x(t+h)\|_p-\| D_{PFp}(A)\,x(t)\|_p)/h=\\
&\lim_{h\downarrow 0}(\| D_{PFp}(A)e^{Ah}x(t)\|_p-\| D_{PFp}(A)x(t)\|_p)/h\leq\\
&\left[\lim_{h\downarrow 0}(\| D_{PFp}(A)\,e^{Ah}\ D_{PFp}^{-1}(A)\|_p-1)/h\right]\| D_{PFp}(A)x(t)\|_p=\\
&\mu_{\|\,\|_p}(D_{PFp}(A)\,A\,D_{PFp}^{-1}(A))\ \| D_{PFp}(A)\,x(t)\|_p=\\
&\lambda_{PF}(A)\,\| D_{PFp}(A)\,x(t)\|_p\,.
\end{aligned}$$

(b) $\Rightarrow$ (a): First we write (4-CT) in the equivalent form

$$\| D_{PFp}(A)\,x(t_0+\tau)\|_p\leq e^{\lambda_{PF}(A)\tau}\,\| D_{PFp}(A)\,x(t_0)\|_p\,,\quad \forall\,\tau,t_0\in\mathbf{R}_+\,, \qquad (4\text{-CT'})$$

from which we get

$$\begin{aligned}
&\| D_{PFp}(A)e^{A\tau}D_{PFp}^{-1}(A)\|_p=\\
&\sup_{x_0\neq 0}\frac{\|(D_{PFp}(A)\,e^{A\tau}D_{PFp}^{-1}(A))\ D_{PFp}(A)x_0\|_p}{\| D_{PFp}(A)x(t_0)\|_p}=\\
&\sup_{x_0\neq 0}\frac{\| D_{PFp}(A)\,x(t_0+\tau;t_0,x_0)\|_p}{\| D_{PFp}(A)x(t_0)\|_p}\leq\ e^{\lambda_{PF}(A)\tau}.
\end{aligned}$$

This yields $\mu_{\| \|_p}(D_{PFp}(A)\, A\, D_{PFp}^{-1}(A)\,) =$ $\lim_{h \downarrow 0}(\|D_{PFp}(A)\, e^{Ah} D_{PFp}^{-1}(A)\|_p - 1)/h \le \lim_{h \downarrow 0}(e^{\lambda_{PF}(A)h} - 1)/h =$ $\lambda_{PF}(A)$, which actually means the equality (3-CT), since the measure of a matrix cannot be less than the PF eigenvalue (e.g. [14]).

(b) $\Rightarrow$ (c): We construct a proof by contradiction, assuming that the time-dependent sets $X_{PFp}^{\varepsilon}(t;t_0)$ defined by (5-CT) are not positively invariant w.r.t. the state-space trajectories of system (1-CT). This means there exist a trajectory $\hat{x}(t) = \hat{x}(t;t_0,x_0)$ of (1-CT), a constant $\varepsilon > 0$, and two moments, $t^*, t^{**} \in \mathbf{R}_+$, $t^{**} > t^* \ge t_0$, such that $\| D_{PFp}(A)\hat{x}(t^*) \|_p \le \varepsilon\, e^{\lambda_{PF}(A)(t^*-t_0)}$ and $\| D_{PFp}(A)\hat{x}(t^{**}) \|_p > \varepsilon\, e^{\lambda_{PF}(A)(t^{**}-t_0)}$. Hence, we get $e^{\lambda_{PF}(A)(t^{**}-t^*)} \| D_{PFp}(A)\,\hat{x}(t^*) \|_p < \| D_{PFp}(A)\ \hat{x}(t^{**}) \|_p$, which contradicts (4-CT').

(c) $\Rightarrow$ (b): By taking $\varepsilon = \| D_{PFp} x_0 \|_p$ in (5-CT), if $\varepsilon > 0$ we get (4-CT'). If $\varepsilon = 0$, then $x_0 = 0$ and hence $x(t) = e^{A(t-t_0)} x_0 = 0$, for all $t \ge t_0$. □

Corollary 2. Let $1 \le p \le \infty$. Let matrix A be essentially nonnegative and irreducible or essentially positive. The following three statements are equivalent.

(a) System (1-CT) is stable.

(b) The function defined by

$$V_{PFp}(x) = \| D_{PFp}(A)\ x \|_p, \tag{6-CT}$$

is a strong Lyapunov function for system (1-CT), with the decreasing rate $\lambda_{PF}(A)$.

(c) The invariant sets defined by (5-CT) are contractive, with the coefficient $\lambda_{PF}(A)$.

Proof: It results from Theorem 2, by taking into account the stability of the matrix A. □

Remark 2. For CT systems, statement (b) of Corollary 2 brings the same generalization as Corollary 1 for DT systems with regard to the construction of diagonal Lyapunov functions by using any Hőlder norm. By considering the classical diagonal-type Lyapunov function $V(x) = x^T \mathbf{\Delta} x$, with $\mathbf{\Delta} = D_{PF2}^2$, and $V_{PF2}(x) = \| D_{PF2}(A)x \|_2$, defined by (6-CT) for $p = 2$, it is obvious that along each trajectory of system (1-CT) we have $D_t^+ V(x) = 2V_{PF2}(x) D_t^+ V_{PF2}(x)$, i.e. $\operatorname{sign} D_t^+ V(x) = \operatorname{sign} D_t^+ V_{PF2}(x)$, $\forall t \in \mathbf{R}_+$. At the same time, the matrix $\mathbf{\Delta} = D_{PF2}^2(A)$ is a solution to the Lyapunov inequality $A^T \mathbf{\Delta} + \mathbf{\Delta} A \prec 0$, where "$\prec 0$" means "negative definite". □

Remark 3. Statement (c) of Corollary 2 generalizes for arbitrary Hőlder p-norms the contractive sets discussed in ([13], Theorem 7.4) only for $p = 1$ and $p = \infty$. □

D. *Comments on the Perron-Frobenius p-Eigenpattern*

The results formulated by the previous two subsections show that the development of our insight into the connections between the PF theory and the linear system dynamics relies on equalities (3-DT) and (3-CT). These equalities hold true if the diagonal matrix $D_{PFp}(A)$ is multiplied by an arbitrary positive constant. This means that the construction of $D_{PFp}(A)$ by procedure (2) does not require the normalization $\sum_{i=1}^{n} r_i = 1$, $\sum_{i=1}^{n} l_i = 1$, of the PF right and left eigenvectors (which was introduced in the first subsection for ensuring the uniqueness of $D_{PFp}(A)$ at that step of the exposition). As expected, the valuable information for our approach is the direction of the PF eigenvectors, since the invariant sets and Lyapunov functions are uniquely defined up to a scaling factor.

III. Dynamics Defined by (Essentially) Nonnegative and Reducible Matrices

In both DT and CT cases, when matrix A is reducible, the right and left eigenvectors associated with the PF eigenvalue may contain 0 elements. Hence, the definition of the matrix $D_{PFp}(A)$ by (2) does not hold any longer. However, we can still use the information given by the PF Theorem if, instead of A, we consider a slightly modified matrix:

$$A_c = A + cU,\ c > 0,\ U = [u_{ij}],\ u_{ij} = 1,\ i, j = 1, \ldots, n, \tag{7}$$

where c is small enough. The matrix A_c is (essentially) positive and we can benefit from the results presented in the previous section. Thus the PF p-eigenpattern of A_c is defined by the pair $\lambda_{PF}(A_c)$ and $D_{PFp}(A_c)$

A. *Properties of Discrete-Time Systems*

In the current subsection, we study the connections between the PF p-eigenpattern of A_c and the properties of the dynamical system (1-DT).

Theorem 3. Let $1 \le p \le \infty$. Let A be nonnegative and reducible. For any $c > 0$, the following three statements are true. Moreover, these statements are equivalent.

(a) $\lambda_{PF}(A) \le \| D_{PFp}(A_c)\, A\, D_{FPp}^{-1}(A_c) \|_p \le \lambda_{PF}(A_c)$. (8-DT)

(b) Along any trajectory $x(t) = x(t;\, t_0, x_0)$ of system (1-DT), the motion fulfills the condition:

$$\| D_{PFp}(A_c)\, x(t+1) \|_p \le \lambda_{PF}(A)\, \| D_{PFp}(A_c)\, x(t) \|_p,\quad \forall t \in \mathbf{Z}_+,\ t \ge t_0. \tag{9-DT}$$

(c) The time-dependent sets

$$X_{PFp}^{\varepsilon}(t;t_0) = \left\{ x \in \mathbf{R}^n \,\middle|\, \|D_{PFp}(A_c)\, x \|_p \le \varepsilon \left(\lambda_{PF}(A_c)\right)^{(t-t_0)} \right\},\quad t, t_0 \in \mathbf{Z}_+,\ t \ge t_0,\ \varepsilon > 0, \tag{10-DT}$$

are (positively) invariant w.r.t. the state-space trajectories of system (1-DT).

Proof. Let us show that (8-DT) is true for the considered hypothesis. We first prove that $\|D_{PFp}(A_c)\,A\,D^{-1}_{PFp}(A_c)\|_p \le \|D_{PFp}(A_c)\,A_c\,D^{-1}_{PFp}(A_c)\|_p$. We can write the componentwise inequalities $|(D_{PFp}(A_c)\,A\,D^{-1}_{PFp}(A_c))\,y| \le |(D_{PFp}(A_c)\,A\,D^{-1}_{PFp}(A_c))|\,|y|| \le |(D_{PFp}(A_c)\,A_c\,D^{-1}_{PFp}(A_c))|\,|y||$, for any $y \in \mathbf{R}^n$.

From Theorem 5.5.10 in [1], the monotonicity of the vector p-norms implies $\|(D_{PFp}(A_c)\,A\,D^{-1}_{PFp}(A_c))\,y\|_p \le$ $\|(D_{PFp}(A_c)\,A\,D^{-1}_{PFp}(A_c))|\,y\,|\,\|_p \le$ $\|(D_{PFp}(A_c)\,A_c\,D^{-1}_{PFp}(A_c))|\,y\,|\,\|_p$, yielding $\|(D_{PFp}(A_c)\,A\,D^{-1}_{PFp}(A_c))\;y\|_p \le$ $\|D_{PFp}(A_c)\,A_c\,D^{-1}_{PFp}(A_c)\|_p\;\|\,|\,y\,|\,\|_p \le$ $\|D_{PFp}(A_c)\,A_c\,D^{-1}_{PFp}(A_c)\|_p\;\|y\|_p$.

Consequently,
$\|D_{PFp}(A_c)\,A\,D^{-1}_{PFp}(A_c)\|_p =$
$\max_{\|y\|_p=1}\|(D_{PFp}(A_c)\,A\,D^{-1}_{PFp}(A_c))\,y\|_p \le \|D_{PFp}(A_c)\,A_c\,D^{-1}_{PFp}(A_c)\|_p$

Now, from the matrix norm properties we have $\lambda_{PF}(A) \le \|D_{PFp}(A_c)\,A\,D^{-1}_{PFp}(A_c)\|_p$ and, on the other hand, from Theorem 1 in [15] we can write $\|D_{PFp}(A_c)\,A_c\,D^{-1}_{PFp}(A_c)\|_p = \lambda_{PF}(A_c)$. Thus (8-DT) is proved. Along the lines of the proof of Theorem 1, we can show that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). □

Corollary 3. Let matrix A be nonnegative and reducible. The following three statements are equivalent.

(a) System (1-DT) is stable.

(b) For any $1 \le p \le \infty$, there exists $c > 0$ such that the function defined by

$$V_{PFp}(x) = \|D_{PFp}(A_c)x\|_p, \qquad (11\text{-DT})$$

is a strong Lyapunov function for system (1-DT), with the decreasing rate as close to $\lambda_{PF}(A)$ as we want.

(c) For any $1 \le p \le \infty$, there exists $c > 0$ such that the invariant sets defined by (10-DT) are contractive, with the coefficient as close to $\lambda_{PF}(A)$ as we want.

Proof: It results from Theorem 3, by taking into account the Schur stability of A and the continuity of $\lambda_{PF}(A_c)$ as a function of $c > 0$, with $\lambda_{PF}(A_c)|_{c=0} = \lambda_{PF}(A)$. □

B. Properties of Continuous-Time Systems

In the current subsection, we study the connections between the PF p-eigenpattern of A_c and the properties of the dynamical system (1-CT).

Theorem 4. Let $1 \le p \le \infty$. Let matrix A be essentially nonnegative and reducible. For any $c > 0$, the following three statements are true. Moreover, these statements are equivalent.

(a) $\lambda_{PF}(A) \le \mu_{\|\,\|_p}(D_{PFp}(A_c)AD^{-1}_{FPp}(A_c)) \le \lambda_{PF}(A_c)$. (8-CT)

(b) Along any trajectory $x(t) = x(t;t_0,x_0)$ of system (1-CT), the motion fulfills the condition:

$$\mathrm{D}_t^+\|D_{PFp}(A_c)x(t)\|_p \le \lambda_{PF}(A_c)\|D_{PFp}(A_c)x(t)\|_p,\quad \forall t \in \mathbf{R}_+,\ t \ge t_0. \qquad (9\text{-CT})$$

(c) The time-dependent sets

$$X^{\varepsilon}_{PFp}(t;t_0) = \left\{x \in \mathbf{R}^n \;\middle|\; \|D_{PFp}(A_c)\;x\|_p \le \varepsilon\, e^{\lambda_{PF}(A_c)(t-t_0)}\right\},\quad t,t_0 \in \mathbf{R}_+,\ t \ge t_0,\ \varepsilon > 0, \qquad (10\text{-CT})$$

are (positively) invariant w.r.t. the state-space trajectories of system (1-CT).

Proof. Let us show that (8-CT) is true for the considered hypothesis. We first prove that $\mu_{\|\,\|_p}(D_{PFp}(A_c)\,A\,D^{-1}_{PFp}(A_c)) \le \mu_{\|\,\|_p}(D_{PFp}(A_c)\,A_c\,D^{-1}_{PFp}(A_c))$

For small $h > 0$ we can write the componentwise inequalities
$0 \le I + hD_{PFp}(A_c)\,A\,D^{-1}_{PFp}(A_c) \le I + hD_{PFp}(A_c)\,A_c\,D^{-1}_{PFp}(A_c)$
that implies
$\|I + h\,D_{PFp}(A_c)\,A\,D^{-1}_{PFp}(A_c)\|_p \le$
$\|I + h\,D_{PFp}(A_c)\,A_c\,D^{-1}_{PFp}(A_c)\|_p$, according to the first part of the proof of Theorem 3. Hence, we get the inequality formulated above, which, together with $\lambda_{PF}(A) \le \mu_{\|\,\|_p}(D_{PFp}(A_c)\,A\,D^{-1}_{PFp}(A_c))$ and $\mu_{\|\,\|_p}(D_{PFp}(A_c)\,A_c\,D^{-1}_{PFp}(A_c)) = \lambda_{PF}(A_c)$, show that (8-CT) is true. Along the lines of the proof of Theorem 2, we can show that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). □

Corollary 4. Let matrix A be essentially nonnegative and reducible. The following three statements are equivalent.

(a) System (1-CT) is stable.

(b) For any $1 \le p \le \infty$, there exists $c > 0$ such that the function defined by

$$V_{PFp}(x) = \|D_{PFp}(A_c)x\|_p, \qquad (11\text{-CT})$$

is a strong Lyapunov function for system (1-CT), with the decreasing rate as close to $\lambda_{PF}(A)$ as we want.

(c) For any $1 \le p \le \infty$, there exists $c > 0$ such that the invariant sets defined by (10-CT) are contractive, with the coefficient as close to $\lambda_{PF}(A)$ as we want.

Proof: It results from Theorem 4, by taking into account the Hurwitz stability of A and the continuity of $\lambda_{PF}(A_c)$ as a function of $c > 0$, satisfying $\lambda_{PF}(A_c)|_{c=0} = \lambda_{PF}(A)$. □

CONCLUSIONS

Our paper reveals important links between the PF eigenstructure and the invariant sets with respect to linear system dynamics. We prove that, for any $1 \le p \le \infty$, the concept of PF p-eigenpattern (which has a purely algebraic nature) induces a family of flow-invariant sets defined by the Hölder vector p-norm. Thus, our results bring a substantial generalization of some properties known in the dynamics of positive linear systems. The analysis has been developed at two levels of complexity that cover types of linear dynamics generated by different classes of matrices.

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APPENDIX

A. *Perron-Frobenius Theorem for positive matrices*

Let $A = [a_{ij}]$ be a real $n \times n$ matrix with positive entries $a_{ij} > 0$, $i, j = 1, \ldots, n$. Then the following statements hold:

1. there is a positive real eigenvalue $\lambda_{PF}(A)$ of A such that any other eigenvalue λ satisfies $|\lambda| < \lambda_{PF}(A)$. (This property may also be stated more concisely by saying that the spectral radius of A is an eigenvalue.)
2. the eigenvalue $\lambda_{PF}(A)$ is simple: $\lambda_{PF}(A)$ is a simple root of the characteristic polynomial of A. In particular, both the right and left eigenspace associated to $\lambda_{PF}(A)$ are 1-dimensional.
3. there is a left and, respectively, a right eigenvector associated with $\lambda_{PF}(A)$ having positive entries. This means that there exist $r = [r_1 \ldots r_n]^T > 0$ and $l = [l_1 \cdots l_n]^T > 0$ such that $Ar = \lambda_{PF}(A)r$ and $A^T l = \lambda_{PF}(A) l$.
4. No other eigenvalue has a positive eigenvector.
5. there exists the eigenvalue estimate $\min_i \sum_j a_{ij} \le \lambda_{PF}(A) \le \max_i \sum_j a_{ij}$.
6. The spectral radius is a strictly increasing function of the matrix entries.

The Perron–Frobenius theorem can be further generalized to the class of block-indecomposable (also called "irreducible") nonnegative matrices. In particular it also holds if some positive power $B = A^k$, $k > 0$ of the nonnegative matrix A has positive entries (matrix A is called "regular" or "primitive").

B. *Perron-Frobenius Theorem for nonnegative matrices*

Let $A = [a_{ij}]$ be a real $n \times n$ matrix with nonnegative entries $a_{ij} \ge 0$, $i, j = 1, \ldots, n$. Then the following statements hold:

1. there is a real eigenvalue $\lambda_{PF}(A)$ of A such that any other eigenvalue λ satisfies $|\lambda| \le \lambda_{PF}(A)$. (This property may also be stated more concisely by saying that the spectral radius of A is an eigenvalue.)
2. there is a left and, respectively, a right eigenvector associated with $\lambda_{PF}(A)$ having nonnegative entries. This means that there exist $r = [r_1 \ldots r_n]^T \ge 0$ and $l = [l_1 \cdots l_n]^T \ge 0$ such that $Ar = \lambda_{PF}(A)r$ and $A^T l = \lambda_{PF}(A) l$.
3. there exists the eigenvalue estimate $\min_i \sum_j a_{ij} \le \lambda_{PF}(A) \le \max_i \sum_j a_{ij}$.

With respect to the theorem concerning positive matrices, in the case of a nonnegative matrix A, the left and right eigenvectors associated with its Perron root $\lambda_{PF}(A)$ are no longer guaranteed to be positive; but they remain non-negative. Furthermore, the Perron root is no longer necessarily simple.

If one requires the matrix A to be irreducible as well as nonnegative, then the results above given for the case of a positive matrix apply. A nonnegative matrix is irreducible if its underlying graph is strongly connected. Note that a positive matrix is always irreducible (as its associated graph is strongly connected), but the converse statement is not necessarily true.