Лаб.6. Оцінка області притягання нелінійної динамічної системи

5. Estimate the region of attraction by using Lyapunov function.

Note that a classical problem of Lyapunov theory is the determination of a domain of attraction [8]. The function $V(x) = x^T Px$ can be used to estimate the region of attraction. Suppose $\dot{V}(x) < 0$, $0 < \|x\| < r$. Letting $c = \min_{\|x\|=r} x^T Px = \lambda_{\min}(P) r^2$ we obtain $\{x^T Px < c\} \subset \{\|x\| < r\}$. This means that all trajectories beginning in the set $\{x^T Px < c\}$ approach the origin as $t \to \infty$. Hence, the set $\{x^T Px < c\}$ is a subset of the region of attraction.

Example 4 Consider the autonomous pendulum with friction

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2 \end{aligned} \tag{21}$$

where $a,\ b > 0$. The Jacobian matrix A at the equilibrium point $x = 0$ is given by:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \tag{22}$$

Then the eigenvalues of A are $(-1 \pm j\sqrt{3})/2$. Hence, the origin is asymptotically stable. Taking $Q = I$ the Lyapunov equation becomes

$$PA + A^T P = -I \tag{23}$$

Solving (23) we obtain

$$P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix} \tag{24}$$

and $\lambda_{\min}(P) = 0.691$. Thus, the system (17) has a candidate Lyapunov function

$$\begin{aligned} V(x) &= x^T Px \\ &= 1.5x_1^2 - x_1x_2 + x_2^2 \end{aligned} \tag{25}$$

We obtain the derivative of $V(x)$ as

$$\begin{aligned} \dot{V}(x) &= (3x_1 - x_2)(-x_2) + (-x_1 + 2x_2)[(x_1 + (x_1^2 - 1)x_2)] \\ &= -(x_1^2 + x_2^2) - (x_1^3x_2 - 2x_1^2x_2^2) \\ &\le -\|x\|^2 + |x_1||x_1x_2||x_1 - 2x_2| \\ &\le -\|x\|^2 + \frac{\sqrt{5}}{2}\|x\|^4 \end{aligned} \tag{26}$$

where $|x_1| \le \|x_1\|$, $|x_1x_2| \le \frac{1}{2}\|x\|^2$, $|x_1 - 2x_2| \le \sqrt{5}\|x\|$. Thus, we obtain

$$\dot{V}(x) < 0, \quad 0 < \|x\|^2 < \frac{2}{\sqrt{5}}$$