Л.7. **Задачі синтезу керування нелінійними динамічними системами, функція керування Ляпунова**

Design of Nonlinear Dynamics Using Feedback

In most of the text we will rely on linear approximations to design feedback laws that stabilize an equilibrium point and provide a desired level of performance. However, for some classes of problems the feedback controller must be nonlinear to accomplish its function. By making use of Lyapunov functions we can often design a nonlinear control law that provides stable behavior, as we saw in Example 4.12.

One way to systematically design a nonlinear controller is to begin with a candidate Lyapunov function $V(x)$ and a control system $\dot{x} = f(x, u)$. We say that $V(x)$ is a *control Lyapunov function* if for every x there exists a u such that $\dot{V}(x) = \frac{\partial V}{\partial x} f(x, u) < 0$. In this case, it may be possible to find a function $\alpha(x)$ such that $u = \alpha(x)$ stabilizes the system. The following example illustrates the approach.

0.4 Using feedback to design stabilizing control

Consider systems of the form $\dot{x} = f(x, u)$. We will not investigate in-depth topics such as Input-to-State-Stability (ISS) and Input-Output Stability (IOS). Instead we will study how control is used to obtain desired stability as pertitent to applications in robotics.

At a basic level, our goal is to obtain u in a feedback-form, i.e.

$$u = \phi(x),$$

so that the resulting *closed-loop* systems has the dynamics

$$\dot{x} = f(x, \phi(x))$$

Example 7. *1-d examples.* Consider the system

$$\dot{x} = ax^2 - x^3 + u, \text{ for some } a \neq 0$$

The simplest approach is to set

$$u = -ax^2 + x^3 - x$$

which results in the closed-loop system

$$\dot{x} = -x$$

which is exponentially stable. This approach was to simply cancel all nonlinear terms. But actually, it is not really necessary to cancel the term $-x^3$ since it is already dissipative. A more economical control law would have just been:

$$u = -ax^2 - x$$

The question of determining a proper u also comes down to finding a Lyapunov function. One approach is to actually specify the Lyapunov function V and a negative definite $\dot{V}$ and then find u to match these choices. For instance, in the example above, let

$$V(x) = \frac{1}{2}x^2$$

and let

$$\dot{V} = ax^3 - x^4 + xu \leq -L(x),$$

for some positive definite $L(x)$. One choice is $L(x) = x^2$ which results in

$$u = -ax^2 + x^3 - x,$$

i.e. the same expensive control law. But another choice is to include higher-order terms, i.e. $L(x) = x^2 + x^4$. Then we have

$$u = -ax^2 - x,$$

which is the preffered control law to globally asymptotically stabilize the system.

Next consider the the trajectory tracking of standard fully-actuated robotic systems. The dynamics is given by

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + N(q, \dot{q}) = u$$

and the task is to track a desired trajectory $q_d(t)$ which is at least twice differentiable. The computed torque law is given by

$$u = M(q)(\ddot{q}_d - K_d\dot{e} - K_p e) + C(q,\dot{q})\dot{q} + N(q,\dot{q}),$$

where $e = q - q_d$ and K_p and K_d are constant matrices. When we substitute this control law we have the following error dynamics

$$\ddot{e} + K_d\dot{e} + K_p e = 0.$$

Since this is a linear equation it is easy to choose K_d and K_p to guarantee that the system is exponentially stable.

Theorem 7. *Stability of computed torque law.* If $K_p, K_d \in \mathbb{R}^{n\times n}$ are positive definite symmetric matrices, then the computed torque law results in exponential trajectory tracking.

Proof: We have the dynamics

$$\frac{d}{dt}\begin{pmatrix} e \\ \dot{e} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ -K_p & -K_d \end{bmatrix}}_{\triangleq A}\begin{pmatrix} e \\ \dot{e} \end{pmatrix}$$

We can show that the eigenvalues of A have negative real parts. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A with corresponding eigenvector $v = (v_1, v_2) \in \mathbb{C}^{2n}$, $v \neq 0$. Then

$$\lambda\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ -K_p & -K_d \end{bmatrix}}_{\triangleq A}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ -K_p v_1 - K_v v_2 \end{pmatrix},$$

which means that if $\lambda = 0$ then $v = 0$ and so $\lambda = 0$ is not an eigenvalue. Similarly, $v_1, v_2 \neq 0$ and we may assume that $\|v_1\| = 1$. Then we have

$$\begin{aligned}\lambda^2 &= v_1^*\lambda^2 v_1 = v_1^*\lambda v_2 \\ &= v_1^*(-K_p v_1 - K_d v_2) = -v_1^* K_p v_1 - \lambda v_1^* K_d v_1,\end{aligned}$$

where $*$ denotes complex conjugate transpose. Since $\alpha \triangleq v_1^* K_p v_1 > 0$ and $\beta \triangleq v_1^* K_d v_1 > 0$ we have

$$\lambda^2 + \alpha\lambda + \beta = 0, \qquad \alpha, \beta > 0,$$

the real part of λ must be negative. □

This is an example of a more general technique known as *feedback linearization*. In subsequent lectures we will generalize these results to underactuated or constrained systems.

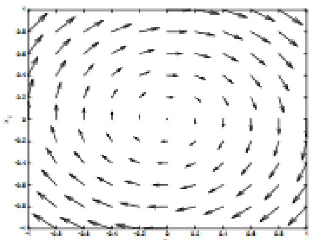
Example: $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_1 u \quad$ and $\quad V(x) = \frac{1}{2}(x_1^2 + x_2^2).$

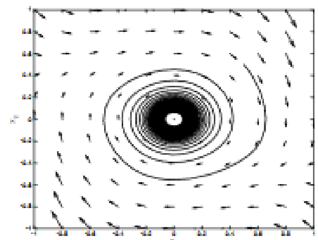
$$\dot{V} = x_1x_2 - x_1x_2 + x_1x_2u \quad \Rightarrow \quad u = -L_gV(x) = -x_1x_2$$

Jurdjevic-Quinn (Nonlinear Damping) Control: If V is such that $L_f V \leq 0$, then $u = -L_g V$ globally asymptotically stabilizes the origin.

Example: $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_1 u$ and $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

$$\dot{V} = x_1 x_2 - x_1 x_2 + x_1 x_2 u \quad \Rightarrow \quad u = -L_g V(x) = -x_1 x_2$$





V. Jurdjevic and J. P. Quinn, "Controllability and Stability", *J. Diff. Eqs.*, 1978.

Example:

$$\dot{x}_1 = -x_1^3 + x_2 \phi(x_1, x_2),$$
$$\dot{x}_2 = \psi(x_1, x_2) + u.$$

$$V(x_1, x_2) = \tfrac{1}{2}(x_1^2 + x_2^2), \; u = -x_2 - \phi(x_1, x_2) - \psi(x_1, x_2) x_1 \Rightarrow \dot{V} = -x_1^4 - x_2^2$$