Л.8. **Метод backstepping стабілізації нелінійних динамічних систем**

A.2.4 Backstepping Technique

The backstepping is a recursive procedure for the construction of nonlinear control laws and Lyapunov functions that guarantee the stability of the latter. This technique is only applicable to a certain class of system which is said to be in strict feedback form (lower triangular). A quick review of this control design approach is given below, see [41] for more details.

Consider the problem of the stabilization of nonlinear systems in the following triangular form:

$$\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_1) \\
\dot{x}_2 &= x_3 + f_2(x_1, x_2) \\
&\vdots \\
\dot{x}_i &= x_{i+1} + f_i(x_1, x_2, \ldots, x_i) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, , x_2, \ldots, x_n) + u
\end{aligned} \tag{A.26}$$

The idea behind the backstepping technique is to consider the state x_2 as a "virtual control " for x_1. Therefore, if it is possible to realize $x_2 = -x_1 - f_1(x_1)$, then the state x_1 will be stabilized. This can be verified by considering the Lyapunov function $V_1 = \frac{1}{2}x_1^2$. However, since x_2 is not the real control for x_1, we make the following change of variables:

$$\begin{aligned}
z_1 &= x_1 \\
z_2 &= x_2 - \alpha_1(x_1)
\end{aligned}$$

with $\alpha_1(x_1) = -x_1 - f_1(x_1)$. By introducing the Lyapunov function $V_1(z_1) = \frac{1}{2}z_1^2$, we obtain

$$\begin{aligned}
\dot{z}_1 &= -z_1 + z_2 \\
\dot{z}_2 &= x_3 + f_2(x_1, x_2) - \frac{\partial \alpha_1}{\partial x_1}\big(x_2 + f_1(x_1)\big) := x_3 + \bar{f}_2(z_1, z_2) \\
\dot{V}_1 &= -z_1^2 + z_1 z_2
\end{aligned}$$

By proceeding recursively, we define the following variables:

$$z_3 = x_3 - \alpha_2(z_1, z_2)$$
$$V_2 = V_1 + \frac{1}{2}z_2^2$$

In order to determine the expression of $\alpha_2(z_1, z_2)$, one can observe that

$$\dot{z}_2 = z_3 + \alpha_2(z_1, z_2) + \bar{f}_2(z_1, z_2)$$
$$\dot{V}_2 = -z_1^2 + z_2\big(z_1 + z_3 + \alpha_2(z_1, z_2) + \bar{f}_2(z_1, z_2)\big)$$

By choosing $\alpha_2(z_1, z_2) = -z_1 - z_2 - \bar{f}_2(z_1, z_2)$, we obtain

$$\dot{z}_1 = -z_1 + z_2$$
$$\dot{z}_2 = -z_1 - z_2 + z_3$$
$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2 z_3$$

Proceeding recursively, at step i, and defining

$$z_{i+1} = x_{i+1} - \alpha_i(z_1, \dots, z_i)$$
$$V_i = \frac{1}{2}\sum_{k=1}^{i} z_k^2$$

we obtain

$$\dot{z}_i = z_{i+1} + \alpha_i(z_1, \dots, z_i) + \bar{f}_i(z_1, \dots, z_i)$$
$$\dot{V}_i = -\sum_{k=1}^{i-1} z_k^2 + z_{i-1}z_i + z_i\big(z_{i+1} + \alpha_i(z_1, \dots, z_i) + \bar{f}_i(z_1, \dots, z_i)\big)$$

By using the expression $\alpha_i(z_1, \dots, z_i) = -z_{i-1} - z_i - \bar{f}_i(z_1, \dots, z_i)$, we obtain

$$\dot{z}_i = -z_{i-1} - z_i + z_{i+1}$$
$$\dot{V}_i = -\sum_{k=1}^{i-1} z_k^2 + z_i z_{i-1}$$

At step n, we obtain

$$\dot{z}_n = \bar{f}_n(z_1, \dots, z_n) + u$$

Choosing

$$u = \alpha_n(z_1, \dots, z_n) = -z_{n-1} - z_n - \bar{f}_n(z_1, \dots, z_n)$$

At step n, we obtain

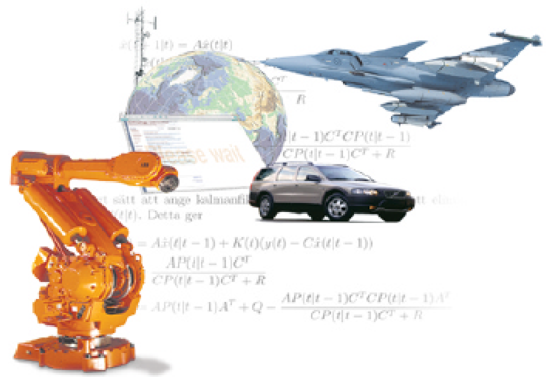
$$\dot{z}_n = \bar{f}_n(z_1, \ldots, z_n) + u$$

Choosing

$$u = \alpha_n(z_1, \ldots, z_n) = -z_{n-1} - z_n - \bar{f}_n(z_1, \ldots, z_n)$$

Backstepping

From simple designs to take-off



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Example 1 (backstepping)

$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = u$$

Step 1:

$$x_{2,d} = -x_1^2 - x_1$$

$$V_1 = \tfrac{1}{2}x_1^2$$

$$\dot{V}_1 = x_1\dot{x}_1 = -x_1^2 \le 0$$

if $x_2 = x_{2,d}$

Step 2:

$$\tilde{x}_2 = x_2 - x_{2,d} = x_2 + x_1^2 + x_1$$

$$\begin{cases} \dot{x}_1 = -x_1 + \tilde{x}_2 \\ \dot{\tilde{x}}_2 = u + \underbrace{(2x_1 + 1)(-x_1 + \tilde{x}_2)}_{\phi} \end{cases}$$

$$V_2 = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}\tilde{x}_2^2$$

$$\dot{V}_2 = x_1(-x_1 + \tilde{x}_2) + \tilde{x}_2(u + \phi)$$

$$= -x_1^2 + \tilde{x}_2(x_1 + u + \phi)$$

$$= -x_1^2 - \tilde{x}_2^2 \le 0$$

if $u = -x_1 - \phi - \tilde{x}_2$

Example 1 (feedback linearization)

$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = u$$

? Which control law should I choose?

$$y = x_1 = \mathbf{z}_1$$
$$\dot{\mathbf{z}}_1 = x_1^2 + x_2 = \mathbf{z}_2$$
$$\dot{\mathbf{z}}_2 = 2\mathbf{z}_1\mathbf{z}_2 + u$$

$\Downarrow$

$u = -2\mathbf{z}_1\mathbf{z}_2 - k_1\mathbf{z}_1 - k_2\mathbf{z}_2$ gives stability

! Same control law with $k_1 = k_2 = 2$

Example 1 (backstepping)

$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = u$$

Step 1: $x_{2,d} = -x_1^2 - x_1$

$$V_1 = \tfrac{1}{2}x_1^2$$

$$\dot{V}_1 = x_1\dot{x}_1 = -x_1^2 \le 0$$

if $x_2 = x_{2,d}$

Step 2: $\tilde{x}_2 = x_2 - x_{2,d} = x_2 + x_1^2 + x_1$

$$\begin{cases} \dot{x}_1 = -x_1 + \tilde{x}_2 \\ \dot{\tilde{x}}_2 = u + \underbrace{(2x_1 + 1)(-x_1 + \tilde{x}_2)}_{\phi} \end{cases}$$

$$V_2 = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}\tilde{x}_2^2$$

$$\dot{V}_2 = x_1(-x_1 + \tilde{x}_2) + \tilde{x}_2(u + \phi)$$

$$= -x_1^2 + \tilde{x}_2(x_1 + u + \phi)$$

$$= -x_1^2 - \tilde{x}_2^2 \le 0$$

if $u = -x_1 - \phi - \tilde{x}_2$

Example 2 (adaptive backstepping)

$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = u + \theta x_2^2$$

Step 1: $x_{2,d} = -x_1^2 - x_1$

$$V_1 = \tfrac{1}{2} x_1^2$$

$$\dot{V}_1 = x_1 \dot{x}_1 = -x_1^2 \le 0$$

Step 2: $\tilde{x}_2 = x_2 - x_{2,d} = x_2 + x_1^2 + x_1$

$$\begin{cases} \dot{x}_1 = -x_1 + \tilde{x}_2 \\ \dot{\tilde{x}}_2 = u + (2x_1 + 1)(-x_1 + \tilde{x}_2) + \theta x_2^2 \end{cases}$$

$$V_2 = \tfrac{1}{2} x_1^2 + \tfrac{1}{2} \tilde{x}_2^2 + \tfrac{1}{2}(\theta - \hat{\theta})^2$$

$$\dot{V}_2 = -x_1^2 + \tilde{x}_2 (x_1 + u + \phi + \theta x_2^2) - (\theta - \hat{\theta})\dot{\hat{\theta}}$$

$$u = -x_1 - \phi - \tilde{x}_2 - \hat{\theta} x_2^2$$ gives

$$\dot{V}_2 = -x_1^2 - \tilde{x}_2^2 + (\theta - \hat{\theta})\left(\tilde{x}_2 x_2^2 - \dot{\hat{\theta}}\right) \le 0$$

if $\dot{\hat{\theta}} = -\tilde{x}_2 x_2^2$