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On Stabilization of Positive Linear Systems

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Abstract

In this paper, we deals with the stabilization problem of the equilibrium points of positive linear systems for the case $r > 1$, in which such equilibrium points belong to interior of the first orthant. We present a sufficient condition that ensure the existence of an affine stabilizing feedback.

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1 Introduction

Let us consider the following linear system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $\mathbf{x} \in \mathbb{R}^n$ is the state vector and $\mathbf{u} \in \mathbb{R}^r$ is the input vector. When the trajectory of the systems is nonnegative from any nonnegative initial state and any nonnegative input, it is said that the system (1) is positive. The formal meaning of the positive linear system is given in [4]. It is well known that system (1) is positive if and only if A is a Metzler Matrix and $B \in \mathbb{R}_{+}^{n\times r}$[5]. The characterization of positive systems attracts interest because this kind of system appears in the modelling of many processes in various field, e.g. in biology, chemistry and economics, see [3], [5] and therein. In these models state variables represent population, measure, mass, etc., and therefore, they are nonnegative. Many aspects of positive linear systems have been considered by different author. A complete introduction to positive linear systems can be found in [2]. Coxson and Shapiro [1] analyzed structural properties of these systems. Stability of the positive linear system has been studied in [2]. It turns out that system (1) is global asymptotic stable if and only if A is Hurwitz, namely, $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(A)$.

Recently, Leenheer and Aeyels [4] discussed the stabilization of equilibrium points of positive linear systems for the case $r = 1$ in which such equilibrium points belong to the interior of the first orthant. They turn out the existence of an interior equilibrium point implies that the system matrix does not possess eigenvalues in the open right half plane. In addition, they provide necessary and sufficient condition to solve the stabilization problem by means of affine state feedback.

In this paper we generalize the stabilization problem of equilibrium points of positive linear systems for the case $r > 1$ in which such equilibrium points belong to the interior of the first orthant, but with conditions that weaker than the hypothesis Proposition 5 in [4]. We also present examples illustrating the result. For shortand and simplicity, all of the terminology, definitions and symbols in [4] are used.

2 Stabilization of Positive Linear Systems

Leenheer and Aeyels in [4] provide some of important assumptions for the case $r = 1$. Based on what is outlined in [4], we give some assumptions.

Assumption 2.1 *For $r > 1$,*

1. *A is a Metzler matrix and $B \in \mathbb{R}_{+}^{n\times r}$*
2. *There exist an $\bar{\mathbf{x}} \in int\left(\mathbb{R}_{+}^{n}\right)$ and $\bar{\mathbf{u}} \in \mathbb{R}_{+}^{r}$ such that $A\bar{\mathbf{x}} + B\bar{\mathbf{u}} = \mathbf{0}$*
3. *$0 \in \sigma(A)$ and for each $\lambda \in \sigma(A) \setminus \{0\}$ holds that $Re(\lambda) < 0$.*

Assumption 2.1 part 3 implies that the system is not global asymptotic stabil, see Corollary 1 in [4]. Therefore, we need to add an extra input vector to the right hand side of (1), that is

$$\dot{\mathbf{x}} = A\mathbf{x} + B\bar{\mathbf{u}} + G\mathbf{v}, \tag{2}$$

where $G \in \mathbb{R}^{n\times r}$ and $\mathbf{v} \in \mathbb{R}^r_+$. In this new system, $\bar{\mathbf{u}}$ is a vector in which each its component is a fixed scalar, while $\mathbf{v}$ is an extra input vector. For $\mathbf{v} \in \mathbb{R}^r_+$, the extended system (2) is positive if and only if $G \in \mathbb{R}^{n\times r}_+$ [4].

Assumption 2.2 *$G \in \mathbb{R}^{n\times r}_+ \setminus \{O\}$ where O is a $n \times r$ zero matrix.*

In [4], it has been proved that if the system (2) satisfies Assumption 2.1 part 1-3 and Assumption 2.2, then there is no continuous stabilizing positive feedback $\mathbf{v}(\mathbf{x}) : \mathbb{R}^n_+ \to \mathbb{R}^r_+$ such that $\bar{\mathbf{x}}$ is global asymptotic stable for the resulting closed loop system with restricting initial condition $\mathbf{x}_0 \in \mathbb{R}^n_+$. This fact motivates one to weaken the constraint on the sign of the feedback $\mathbf{v}(\mathbf{x})$, that is, the feedback $\mathbf{v}(\mathbf{x})$ need not belong to $\mathbb{R}^r_+$. On the other hand, the resulting closed loop system should remain a positive system.

By following [4], we define an affine feedback $\mathbf{v}(\mathbf{x}) = \bar{K}^T(\mathbf{x} - \bar{\mathbf{x}})$ for some $\bar{K} \in \mathbb{R}^{n\times r}$ such that the resulting closed loop system is

$$\dot{\mathbf{x}} = (A + G\bar{K}^T)\mathbf{x} + (B\bar{\mathbf{u}} - G\bar{K}^T\bar{\mathbf{x}}). \tag{3}$$

Hence, the problem under consideration is if the system (2) satisfies Assumption 2.1 part 1-3 and Assumption 2.2, does there exists some affine feedback $\mathbf{v}(\mathbf{x}) = \bar{K}^T(\mathbf{x} - \bar{\mathbf{x}})$ for some $\bar{K} \in \mathbb{R}^{n\times r}$ such that

1. $(A + G\bar{K}^T)$ is Metzler and $(B\bar{\mathbf{u}} - G\bar{K}^T\bar{\mathbf{x}}) \in \mathbb{R}^n_+$.
2. $(A + G\bar{K}^T)$ is Hurwitz.

The following result is an answer for the this question, it is a weaker version than Proposition 5 in [4]. Its proving process requires Lemma 1 in [4].

Theorem 2.3 *An affine feedback $\bar{\mathbf{v}}(\mathbf{x}) = \bar{K}^T(\mathbf{x} - \bar{\mathbf{x}})$ exists for some $\bar{K} \in \mathbb{R}^{n\times r}_-$ such that $(A + G\bar{K}^T)$ be a Metzler matrix and Hurwitz, and $(B\bar{\mathbf{u}} - G\bar{K}^T\bar{\mathbf{x}}) \in \mathbb{R}^n_+$ if*

1. *equation (2) with $G \in \mathbb{R}^{n\times r}_+$ and $\mathbf{v} \in \mathbb{R}^r_+$ satisfy the Assumption 2.1 part 1-3 and Assumption 2.2,*
2. *there are an affine feedback $\mathbf{v}(\mathbf{x}) = K^T(\mathbf{x} - \bar{\mathbf{x}})$ for some $K \in \mathbb{R}^{n\times r}$ such that $(A + GK^T)$ be a Metzler matrix and $(B\bar{\mathbf{u}} - GK^T\bar{\mathbf{x}}) \in \mathbb{R}^n_+$.*

Proof. Let the system (2) satisfies Assumption 1 part 1-3 and Assumption 2, and there are an affine feedback $\mathbf{v}(\mathbf{x}) = K^T(\mathbf{x} - \bar{\mathbf{x}})$ for some $K \in \mathbb{R}^{n\times r}$ such that $(A + GK^T)$ be a Metzler matrix and $(B\bar{\mathbf{u}} - GK^T\bar{\mathbf{x}}) \in \mathbb{R}^{n\times r}_+$. Define $\bar{K} \in \mathbb{R}^{n\times r}_-$ by

$$\bar{k}_{lm} = \min\{0, k_{lm}\} \text{ for } l = 1, 2, \ldots, n, \quad m = 1, 2, \ldots, r, \tag{4}$$

then $\bar{k}_{lm} \leq k_{lm}$ and $\bar{k}_{lm} \leq 0$ for $l = 1, 2, \ldots, n, \quad m = 1, 2, \ldots, r$. This implies that for each $l \neq j$, the components of the matrix $(A + G\bar{K}^T)$ are given by

$$a_{lj} + \sum_{m=1}^{r} g_{lm}\bar{k}_{jm} = \begin{cases} a_{lj} + \sum_{m=1}^{r} g_{lm}k_{jm}, & \text{if } \sum_{m=1}^{r} g_{lm}k_{jm} \leq 0 \\ a_{lj}, & \text{if } \sum_{m=1}^{r} g_{lm}k_{jm} > 0. \end{cases} \tag{5}$$

Since both A and $(A+GK^T\bar{\mathbf{x}})$ are Metzler matrices, (4) implies that $(A+G\bar{K}^T)$ is a Metzler matrix. Furthermore, $(B\bar{\mathbf{u}}-GK^T\bar{\mathbf{x}}) \in \mathbb{R}_+^n$ implies $(B\bar{\mathbf{u}}-G\bar{K}^T\bar{\mathbf{x}}) \in \mathbb{R}_+^n$. Moreover, Lemma 1 in [4] implies that

$$\lambda_0(A + G\bar{K}^T) < \lambda_0(A) < 0.$$

This fact shows that $(A + G\bar{K}^T)$ is Hurwitz. $\square$

It is obvious that this theorem only assumes the existence of an affine feedback $\mathbf{v}(\mathbf{x}) = K^T(\mathbf{x} - \bar{\mathbf{x}})$ for some $K \in \mathbb{R}^n$ that satisfy $(A + GK^T)$ be a Metzler matrix. Note that if $(A+GK^T)$ be a Metzler matrix then it needs not Hurwitz. Thus, for the case $r = 1$, the hypothesis of Theorem 2.3 is weaker than the hypothesis of Proposition 5 in [4] that assume the existence an affine stabilizing feedback $\mathbf{v}(\mathbf{x}) = K^T(\mathbf{x} - \bar{\mathbf{x}})$ for some $K \in \mathbb{R}^n$.

3 Numerical Examples

The following example illustrates the Proposition 5 in [4].

Example 3.1 *Let us consider the system* (1) *where the matrices A and B are given as follows:*

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

In this example, if $\mathbf{u} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ then $\bar{\mathbf{x}} = \begin{pmatrix} 1.5 & 1 & 1 \end{pmatrix}^T \in int\left(\mathbb{R}_+^3\right)$ is an equilibrium point of the system under consideration. Moreover, $\sigma(A) = \{-2, -3, 0\}$. Furthermore, if given

$$G = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } K = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

then we have

$$\mathbf{v}(\mathbf{x}) = \begin{pmatrix} x_1 - x_2 + x_3 - \frac{3}{2} \\ 1 - x_2 \\ 1 - x_3 \end{pmatrix} \quad \text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Using this $\mathbf{v}(\mathbf{x})$, we have the following new system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1.5 \\ 1 \\ 3 \end{pmatrix}.$$

It is obvious that

$$\left(A + GK^T\right) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

is a Metzler matrix and Hurwitz with $\sigma\left(A + GK^T\right) = \{-3, -1\}$. Therefore, this $\mathbf{v}(\mathbf{x})$ is an affine stabilizing feedback, thus in accordance to Proposition 5 in [4], an affine stabilizing feedback $\bar{\mathbf{v}}(\mathbf{x}) = \bar{K}^T(\mathbf{x} - \bar{\mathbf{x}})$ exists for some $\bar{K} \in \mathbb{R}_-^{n\times r}$. This fact is not opposite to the Theorem 2.3.

The following example illustrates the Theorem 2.3.

Example 3.2 *Consider the following matrices:*

$$A = \begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For $\mathbf{u} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$, we have $\bar{\mathbf{x}} = \begin{pmatrix} 1 & 0.5 & 1 \end{pmatrix}^T \in int\left(\mathbb{R}_+^3\right)$ is an equilibrium point. Moreover, $\sigma(A) = \{-3, 0, -2\}$. Furthermore, if given

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } K = \begin{pmatrix} 1 & 0 \\ 0 & -2 \\ -1 & 0 \end{pmatrix},$$

then we have

$$\mathbf{v}(\mathbf{x}) = \begin{pmatrix} x_1 - x_3 \\ 1 - 2x_2 \end{pmatrix} \text{ where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Using this $\mathbf{v}(\mathbf{x})$, we have the following new system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -4 & 0 \\ 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

It is clear that

$$\left(A + GK^T\right) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -4 & 0 \\ 2 & 0 & -2 \end{pmatrix}$$

is a Metzler matrix but $\sigma\left(A+GK^T\right)=\{-4,0,-3\}$, thus the matrix $\left(A+GK^T\right)$ is not Hurwitz. Choose

$$\bar{K}=\begin{pmatrix} 0 & 0 \\ 0 & -2 \\ -1 & 0 \end{pmatrix},$$

we find

$$\left(A+G\bar{K}^T\right)=\begin{pmatrix} -2 & 0 & 1 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

that is a Metzler matrix and Hurwitz with $\sigma\left(A+G\bar{K}^T\right)=\{-3,-4,-1\}$.

This example explain that the existence of an affine feedback $\mathbf{v}(\mathbf{x})$ such that the matrix $\left(A+GK^T\right)$ is a Metzler is adequate to conclude the existence of an affine feedback $\bar{\mathbf{v}}(\mathbf{x})$ such that the matrix $\left(A+G\bar{K}^T\right)$ is a Metzler and Hurwitz.

4 Conclusion

We have already give a sufficient condition that assure the existence of an affine stabilizing feedback. This condition is weaker than condition of the hypothesis Proposition 5 in [4].

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