(M , β) -Stability of Positive Linear Systems

Article *in* Mathematical Problems in Engineering · February 2016

DOI: 10.1155/2016/9605464

2 authors:

Octavian Pastravanu
Gheorghe Asachi Technical University of Iasi

Mihaela Hanako Matcovschi
Gheorghe Asachi Technical University of Iasi

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2016, Article ID 9605464, 11 pages
http://dx.doi.org/10.1155/2016/9605464

Research Article

(M, β)-Stability of Positive Linear Systems

Octavian Pastravanu and Mihaela-Hanako Matcovschi

Department of Automatic Control and Applied Informatics, Technical University "Gheorghe Asachi" of Iasi, 700050 Iasi, Romania

Correspondence should be addressed to Mihaela-Hanako Matcovschi; mhanako@ac.tuiasi.ro

Received 5 August 2015; Revised 11 January 2016; Accepted 14 January 2016

Academic Editor: Asier Ibeas

The main purpose of this work is to show that the Perron-Frobenius eigenstructure of a positive linear system is involved not only in the characterization of long-term behavior (for which well-known results are available) but also in the characterization of short-term or transient behavior. We address the analysis of the short-term behavior by the help of the "(M, β)-stability" concept introduced in literature for general classes of dynamics. Our paper exploits this concept relative to Hölder vector p-norms, $1 \le p \le \infty$, adequately weighted by scaling operators, focusing on positive linear systems. Given an asymptotically stable positive linear system, for each $1 \le p \le \infty$, we prove the existence of a scaling operator (built from the right and left Perron-Frobenius eigenvectors, with concrete expressions depending on p) that ensures the best possible values for the parameters M and β, corresponding to an "ideal" short-term (transient) behavior. We provide results that cover both discrete- and continuous-time dynamics. Our analysis also captures the differences between the cases where the system dynamics is defined by matrices irreducible and reducible, respectively. The theoretical developments are applied to the practical study of the short-term behavior for two positive linear systems already discussed in literature by other authors.

1. Introduction

1.1. Notation. Let $\mathbf{x} = [x_1 \cdots x_n]^\top \in \mathbb{R}^n$ be a vector. The Hölder vector p-norm is defined as $\|\mathbf{x}\|_p = [|x_1|^p + \cdots + |x_n|^p]^{1/p}$ for $1 \le p < \infty$, and $\|\mathbf{x}\|_\infty = \max_{i=1,\ldots,n}|x_i|$ for $p = \infty$.

Let $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a square matrix. Let $\sigma(\mathbf{A})$ be the *spectrum* of matrix $\mathbf{A}$; the eigenvalues of $\mathbf{A}$ are denoted by $\lambda_i(\mathbf{A}) \in \sigma(\mathbf{A})$, $i = \overline{1,n}$. The norm of matrix $\mathbf{A}$ induced by a vector norm $\|\ \|$ (not necessarily a Hölder p-norm) is defined as $\|\mathbf{A}\| = \max_{\mathbf{x}\in\mathbb{R}^n, \|\mathbf{x}\|=1}\|\mathbf{A}\mathbf{x}\|$. The matrix measure of $\mathbf{A}$ with respect to a matrix norm $\|\ \|$ is given by $\mu_{\|\ \|}(\mathbf{A}) = \lim_{h\searrow 0}(1/h)[\|\mathbf{I} + h\mathbf{A}\| - 1]$, where $\mathbf{I}$ stands for the unit matrix of order n. In the particular case of Hölder p-norms $p \in \{1, 2, \infty\}$, the expressions of the induced matrix norms are $\|\mathbf{A}\|_1 = \max_{j=1,\ldots,n}\sum_{i=1}^n |a_{ij}|$, $\|\mathbf{A}\|_2 = (\lambda_{\max}(\mathbf{A}^T\mathbf{A}))^{1/2}$, $\|\mathbf{A}\|_\infty = \max_{i=1,\ldots,n}\sum_{j=1}^n |a_{ij}|$, and the corresponding matrix measures are $\mu_{\|\ \|_1}(\mathbf{A}) = \max_{j=1,\ldots,n}\{a_{jj} + \sum_{i=1, i\neq j}^n |a_{ij}|\}$, $\mu_{\|\ \|_2}(\mathbf{A}) = 1/2\lambda_{\max}(\mathbf{A} + \mathbf{A}^T)$, $\mu_{\|\ \|_\infty}(\mathbf{A}) = \max_{i=1,\ldots,n}\{a_{ii} + \sum_{j=1, j\neq i}^n |a_{ij}|\}$, as per Fact 11.15.7 from [1].

1.2. Concept of (M, β)-Stability. The concept of (M, β)-*stability* has been developed by the monographic work in [2], aiming to offer a refined characterization of the short-term behavior (also called transient behavior) of the exponentially stable systems, in the sense of the dynamics properties exhibited by the free response.

In particular, the cited work provides adequate instruments for the analysis of both *long-term* and *short-term* behavior of linear systems with discrete-time (abbreviated DT) dynamics,

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{A} \in \mathbb{R}^{n\times n}, \qquad \mathbf{x}(0) = \mathbf{x}_0, \quad t \in \mathbb{Z}_+, \tag{1-DT}$$

and continuous-time (abbreviated CT) dynamics,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{A} \in \mathbb{R}^{n\times n}, \ \mathbf{x}(0) = \mathbf{x}_0, \ t \in \mathbb{R}_+. \tag{1-CT}$$

Throughout the text, we intend to develop a parallel analysis of the DT and CT cases, reason for which the equation numbering includes the extensions -DT and -CT, respectively, as above. We are going to refer concomitantly to equations numbered as (#-DT) and (#-CT) by using the formulation "(#-DT) (resp., (#-CT))". This type of parallel approach will also have a more general formulation, in the sense of "property of DT system (resp., property of CT system)".

The analysis of the *long-term behavior* of system (1-DT) (resp., (1-CT)) relies on the notion of *growth rate*, denoted by $\alpha(\mathbf{A})$, which is defined as in Subsection 3.3.2 of [2], by

$$\alpha(\mathbf{A}) = \lim_{t\to\infty}\left(\left\|\mathbf{A}^t\right\|\right)^{1/t} = \max_{i=1,\dots,n}\left|\lambda_i(\mathbf{A})\right|, \quad \text{(2-DT)}$$

for the DT case, and, respectively,

$$\alpha(\mathbf{A}) = \lim_{t\to\infty}\ln\left(\left\|e^{\mathbf{A}t}\right\|\right)^{1/t} = \max_{i=1,\dots,n}\operatorname{Re}\left\{\lambda_i(\mathbf{A})\right\}, \quad \text{(2-CT)}$$

for the CT case. In algebraic terms, the growth rate equals the *spectral radius* of $\mathbf{A}$ for system (1-DT) and, respectively, the *spectral abscissa* of $\mathbf{A}$ for system (1-CT).

The analysis of the *transient* (*short-term*) *behavior* of system (1-DT) (resp., (1-CT)) relies on the notion of (M,β)-*stability* which is defined as follows.

Definition 1 (Definition 5.5.1 [2]). Let $1 \le M$ and $\alpha(\mathbf{A}) \le \beta$, with $0 < \beta < 1$ in the DT case, and, respectively, $\beta < 0$ in the CT case. Consider an absolute vector norm $\|\ \|$ in the state-space $\mathbb{R}^n$. System (1-DT) (resp., (1-CT)) is said to be (M,β)-stable relative to the norm $\|\ \|$, if its trajectories satisfy the inequality

$$\left\|\mathbf{x}(t;0,\mathbf{x}_0)\right\| \le M\beta^t\left\|\mathbf{x}_0\right\|, \quad \forall\mathbf{x}_0\in\mathbb{R}^n,\ \forall t\in\mathbb{Z}_+, \quad \text{(3-DT)}$$

in the DT case, and, respectively,

$$\left\|\mathbf{x}(t;0,\mathbf{x}_0)\right\| \le Me^{\beta t}\left\|\mathbf{x}_0\right\|, \quad \forall\mathbf{x}_0\in\mathbb{R}^n,\ \forall t\in\mathbb{R}_+, \quad \text{(3-CT)}$$

in the CT case. The scalar M is called the *transient bound* and the scalar β is called the *exponential rate*.

In colloquial terms, a "good" transient behavior relative to norm $\|\ \|$ means M close to 1 and β close to $\alpha(\mathbf{A})$. The *"ideal" transient behavior* relative to norm $\|\ \|$ can be introduced by formal specifications, namely, $M = 1$ and $\beta = \alpha(\mathbf{A})$ (i.e., both minimal exponential rate and minimal transient bound). Obviously, for these particular values of M and β, inequality (3-DT) (resp., (3-CT)) becomes inequality

$$\left\|\mathbf{x}(t;0,\mathbf{x}_0)\right\| \le (\alpha(\mathbf{A}))^t\left\|\mathbf{x}_0\right\|, \quad \forall\mathbf{x}_0\in\mathbb{R}^n,\ \forall t\in\mathbb{Z}_+, \quad \text{(4-DT)}$$

in the case of system (1-DT), and, respectively,

$$\left\|\mathbf{x}(t;0,\mathbf{x}_0)\right\| \le e^{\alpha(\mathbf{A})t}\left\|\mathbf{x}_0\right\|, \quad \forall\mathbf{x}_0\in\mathbb{R}^n,\ \forall t\in\mathbb{R}_+, \quad \text{(4-CT)}$$

in the case of system (1-CT).

Both growth rate and (M,β)-stability are rigorously related to the properties of the semigroup of operators generated by $\mathbf{A}$ (discrete- or continuous-time), for example, [3]. Consider the semigroup of linear operators generated by matrix $\mathbf{A}$. Then, by using the operator norm induced by the vector norm $\|\ \|$, inequality (3-DT) can be equivalently written for operators $\mathbf{A}^t$, $t\in\mathbb{Z}_+$, in the form

$$\left\|\mathbf{A}^t\right\| \le M\beta^t, \quad \forall t\in\mathbb{Z}_+, \quad \text{(5-DT)}$$

and inequality (3-CT) can be equivalently written for operators $e^{\mathbf{A}t}$, $t\in\mathbb{R}_+$, in the form

$$\left\|e^{\mathbf{A}t}\right\| \le Me^{\beta t}, \quad \forall t\in\mathbb{R}_+. \quad \text{(5-CT)}$$

If we take $M = 1$ in inequality (5-DT) (resp., (5-CT)), then *the initial growth rate* relative to the norm $\|\ \|$ (see Definition 5.5.7 and Proposition 5.5.8 in [2]), denoted by $\alpha_{\|\ \|}(\mathbf{A})$, is defined as

$$\alpha_{\|\ \|}(\mathbf{A}) = \min\left\{\beta\in\mathbb{R} \mid \left\|\mathbf{A}^t\right\| \le \beta^t,\ \forall t\in\mathbb{Z}_+\right\} = \|\mathbf{A}\|, \quad \text{(6-DT)}$$

for the DT case, and, respectively, as

$$\begin{aligned}\alpha_{\|\ \|}(\mathbf{A}) &= \min\left\{\beta\in\mathbb{R} \mid \left\|e^{\mathbf{A}t}\right\| \le e^{\beta t},\ \forall t\in\mathbb{R}_+\right\} \\ &= \lim_{h\downarrow 0}\frac{\left\|e^{\mathbf{A}h}\right\|-1}{h} = \lim_{h\downarrow 0}\frac{\|\mathbf{I}+\mathbf{A}h\|-1}{h} \\ &= \mu_{\|\ \|}(\mathbf{A}),\end{aligned} \quad \text{(6-CT)}$$

for the CT case.

By comparing (2-DT) with (6-DT) (resp., (2-CT) with (6-CT)), one can simply notice that the inequality $\alpha(\mathbf{A}) \le \alpha_{\|\ \|}(\mathbf{A})$ holds true regardless of the considered norm. (In algebraic terms, the spectral radius is less than or equal to any matrix norm, and, respectively, the spectral abscissa is less than or equal to any matrix measure.) This fact makes completely clear the difference between the *global sense* of the growth rate and the *local sense* of the initial growth rate. Subsequently, the "ideal" transient behavior relative to norm $\|\ \|$ is characterized by the equality $\alpha(\mathbf{A}) = \alpha_{\|\ \|}(\mathbf{A})$.

Obviously, inequality (4-DT) (resp., (4-CT)) is satisfied by the trajectories of any system (1-DT) (resp., (1-CT)) whose matrix $\mathbf{A}$ is diagonalizable, once the considered vector norm $\|\ \|$ is defined as an absolute norm, weighted by the diagonalizing matrix. This represents a trivial example of the "ideal" transient behavior, with low practical interest, since the state variables of the diagonalized system are linear combinations of the original state variables, and the transient bounds of the former may seldom have a useful meaning for the latter. Therefore, the study of (M,β)-stability provides relevant information only when the considered norm $\|\ \|$ is able to preserve the key role played by the original state-space vector.

In work [2], the "ideal" transient behavior is illustrated by a single class of linear systems, namely, those where matrix $\mathbf{A}$ in equality (1-DT) (resp., (1-CT)) is a *normal matrix*, and the norm $\|\ \|$ considered in inequality (4-DT) (resp., (4-CT)) is the 2-norm.

1.3. Paper Objective and Organization. The main objective of our paper is to expand the analysis framework of (M,β)-stability and "ideal" transient behavior by focusing on the class of positive linear systems, whose dynamics are generated by matrices **A** *nonnegative* in equality (1-DT), and *essentially nonnegative* in equality (1-CT). Literature includes several remarkable monographs on positive systems, among which we mention [4–6], that cover both analysis and design topics, by creating a wide perspective on the structure and behavioral particularities of various types of systems (social, economic, biological, and technical). The connections between the algebraic characterization of (essentially) nonnegative matrices and the dynamical properties of positive linear systems are deeply explored by [7, 8].

Our paper shows that positive linear system exhibits an "ideal" transient behavior relative to any vector p-norm ($1 \leq p \leq \infty$) considered in inequality (4-DT) (resp., (4-CT)), if the state-space variables are individually scaled by appropriate values. Concisely speaking, these scaling values are intimately related to the elements of the left and right Perron-Frobenius eigenvectors of matrix **A**. In norm terms, the use of the scaled state-space variables represents a simple weighting of the standard vector p-norms by positive definite diagonal matrices, fact which does not alter the meaning of the state-space vector components. Our approach includes, as particular cases corresponding to $p \in \{1, 2, \infty\}$, the properties of scaled positive systems presented by the recent paper [9] in Propositions 1 and 2 and Remark 1.

Thus, as an overall comment on the contribution brought by this paper, we notice that, besides the extension of the investigation for (M, β)-stability along the lines proposed by [2], it also reveals deeper connections between the Perron-Frobenius eigenstructure and the dynamics of positive linear systems. Connections of this type are mentioned by many works devoted to positive linear systems, such as [4–14]. Nevertheless, the cited works do not explore the role of the Perron-Frobenius eigenstructure in the characterization of the short-term behavior of positive linear systems.

The remainder of the text is organized as follows. The main results are presented by Section 2, for positive systems defined by irreducible matrices, and by Section 3, for positive systems defined by reducible matrices. Section 4 illustrates the applicability of the theoretical results in studying the short-term dynamics of two positive linear systems previously discussed by other works. Section 5 formulates some concluding remarks on the importance of our research.

2. Results for Positive Systems Defined by Irreducible Matrices

Throughout this section, matrix **A** that defines the dynamics of system (1-DT) (resp., (1-CT)) is *irreducible* (e.g., [8, Chapter 2, Definition 1.2]). Matrix **A** is irreducible if and only if the oriented graph $G(\mathbf{A})$ associated with **A** is strongly connected (e.g., [8, Chapter 2, Theorem 2.7]). In the DT case matrix **A** is *nonnegative* (i.e., all its entries are nonnegative). In the CT case matrix **A** is *essentially nonnegative* or *Metzler* (i.e., all its off-diagonal entries are nonnegative).

The meanings of the Perron-Frobenius eigenstructure agree for **A** nonnegative and **A** essentially nonnegative, in the sense of the following properties:

(i) Matrix **A** has a simple real eigenvalue, called the *Perron-Frobenius eigenvalue* and denoted by $\lambda_{\max}(\mathbf{A})$, which satisfies $|\lambda_i(\mathbf{A})| \leq \lambda_{\max}(\mathbf{A})$, $i = 1, \ldots, n$, in the DT case, and $\operatorname{Re}\{\lambda_i(\mathbf{A})\} \leq \lambda_{\max}(\mathbf{A})$, $i = 1, \ldots, n$, in the CT case.

(ii) In both DT and CT cases, matrix **A** has right and left positive eigenvectors $\mathbf{v} = [v_1 \cdots v_n]^T \gg 0$, $\mathbf{w} = [w_1 \cdots w_n]^T \gg 0$ associated with $\lambda_{\max}(\mathbf{A})$, which are called the right and left *Perron-Frobenius eigenvectors* and satisfy $\mathbf{A}\mathbf{v} = \lambda_{\max}(\mathbf{A})\mathbf{v}$ and $\mathbf{A}^T\mathbf{w} = \lambda_{\max}(\mathbf{A})\mathbf{w}$.

Given the irreducible matrix **A**, for any p, $1 \leq p \leq \infty$, we define the *p-type Perron-Frobenius scaling operator*

$$\mathbf{D}_p = \begin{cases} \operatorname{diag}\{[(w_1)\cdots(w_n)]\}, & \text{if } p = 1, \\ \operatorname{diag}\left\{\left[(w_1)^{1/p}(v_1)^{(-p+1)/p}\cdots(w_n)^{1/p}(v_n)^{(-p+1)/p}\right]\right\}, & \text{if } 1 < p < \infty, \\ \operatorname{diag}\left\{\left[(v_1)^{-1}\cdots(v_n)^{-1}\right]\right\}, & \text{if } p = \infty, \end{cases} \quad \text{(7-DT//CT)}$$

built from the positive eigenvectors $\mathbf{v} \gg 0$, $\mathbf{w} \gg 0$, of **A**, which satisfy $\|\mathbf{v}\|_\infty = 1$, $\|\mathbf{w}\|_\infty = 1$.

Theorem 2. *Let $1 \leq p \leq \infty$. Consider an asymptotically stable, positive linear system of form* (1-DT) *(resp.,* (1-CT)*), with matrix **A** irreducible.*

Then, system (1-DT) *(resp.,* (1-CT)*) is* $(1, \alpha(\mathbf{A}))$*-stable relative to the norm $\mathscr{V}_p(\mathbf{x}) = \|\mathbf{D}_p\mathbf{x}\|_p$, where $\mathbf{D}_p$ is the scaling operator defined by* (7-DT//CT).

Proof. For any $1 \leq p \leq \infty$, $\mathscr{V}_p(\mathbf{x})$ is a vector norm, defined by the standard p-norm, weighted by the scaling operator $\mathbf{D}_p$ (7-DT//CT).

(a) In the DT case, for any $1 \leq p \leq \infty$, for an arbitrary trajectory we can write

$$\begin{aligned} \mathscr{V}_p(\mathbf{x}(t+1)) &= \left\|\mathbf{D}_p\mathbf{x}(t+1)\right\|_p = \left\|\mathbf{D}_p\mathbf{A}\mathbf{x}(t)\right\|_p \\ &= \left\|\mathbf{D}_p\mathbf{A}\mathbf{D}_p^{-1}\mathbf{D}_p\mathbf{x}(t)\right\|_p \\ &\leq \left\|\mathbf{D}_p\mathbf{A}\mathbf{D}_p^{-1}\right\|_p\left\|\mathbf{D}_p\mathbf{x}(t)\right\|_p \\ &= \left\|\mathbf{D}_p\mathbf{A}\mathbf{D}_p^{-1}\right\|_p\mathscr{V}_p(\mathbf{x}(t)). \end{aligned} \quad \text{(8-DT)}$$

On the other hand, we have

$$\left\|\mathbf{D}_p\mathbf{A}\mathbf{D}_p^{-1}\right\|_p = \lambda_{\max}(\mathbf{A}) = \alpha(\mathbf{A}) \tag{9-DT}$$

as resulting from the proof of the theorem presented by paper [15].

Thus, from (8-DT) and (9-DT) we get the inequality

$$\mathscr{V}_p(\mathbf{x}(t+1)) \le \alpha(\mathbf{A})\mathscr{V}_p(\mathbf{x}(t)), \quad \forall t \in \mathbb{Z}_+, \tag{10-DT}$$

and, eventually,

$$\mathscr{V}_p(\mathbf{x}(t;0,\mathbf{x}_0)) \le (\alpha(\mathbf{A}))^t \mathscr{V}_p(\mathbf{x}_0), \quad \forall \mathbf{x}_0 \in \mathbb{R}^n,\ \forall t \in \mathbb{Z}_+, \tag{11-DT}$$

proving that system (1-DT) is (M,β)-stable, with $M = 1$ and $\beta = \alpha(\mathbf{A})$, in accordance with (3-DT).

(b) For the CT case, for any $1 \le p \le \infty$, we can write

$$\begin{aligned}
&\mathscr{D}_t^+\mathscr{V}_p(\mathbf{x}(t)) \\
&= \lim_{h\downarrow 0}\frac{1}{h}\left[\mathscr{V}_p(\mathbf{x}(t+h)) - \mathscr{V}_p(\mathbf{x}(t))\right] \\
&= \lim_{h\downarrow 0}\frac{1}{h}\left[\left\|\mathbf{D}_p\mathbf{x}(t+h)\right\|_p - \left\|\mathbf{D}_p\mathbf{x}(t)\right\|_p\right] \\
&= \lim_{h\downarrow 0}\frac{1}{h}\left[\left\|\mathbf{D}_p e^{\mathbf{A}h}\mathbf{D}_p^{-1}\mathbf{D}_p\mathbf{x}(t)\right\|_p - \left\|\mathbf{D}_p\mathbf{x}(t)\right\|_p\right] \\
&\le \left(\lim_{h\downarrow 0}\frac{1}{h}\left[\left\|\mathbf{D}_p e^{\mathbf{A}h}\mathbf{D}_p^{-1}\right\|_p - 1\right]\right)\left\|\mathbf{D}_p\mathbf{x}(t)\right\|_p \\
&= \left(\mathscr{D}_t^+\left\|e^{\mathbf{D}_p\mathbf{A}\mathbf{D}_p^{-1}t}\right\|_p\Big|_{t=0}\right)\mathscr{V}_p(\mathbf{x}(t)).
\end{aligned} \tag{8-CT}$$

On the other hand, we have

$$\begin{aligned}
&\mathscr{D}_t^+\left\|e^{\mathbf{D}_p\mathbf{A}\mathbf{D}_p^{-1}t}\right\|_p\Big|_{t=0} \\
&= \lim_{h\downarrow 0}\frac{1}{h}\left[\left\|\mathbf{I} + h\mathbf{D}_p\mathbf{A}\mathbf{D}_p^{-1}\right\|_p - 1\right] = \lambda_{\max}(\mathbf{A}) \\
&= \alpha(\mathbf{A}).
\end{aligned} \tag{9-CT}$$

Indeed, the first equality in (9-CT) results from Fact 11.15.7 [1]. For the second equality, let us pick an $s > 0$ such that $s\mathbf{I} + \mathbf{A}$ is nonnegative. Since matrices $s\mathbf{I} + \mathbf{A}$ and $\mathbf{A}$ have the same left and right Perron-Frobenius eigenvectors, by using part (a) of our proof, for any $1 \le p \le \infty$, we get $s + \lambda_{\max}(\mathbf{A}) = \lambda_{\max}(s\mathbf{I} + \mathbf{A}) = \|\mathbf{D}_p(s\mathbf{I} + \mathbf{A})\mathbf{D}_p^{-1}\|_p = \|s\mathbf{I} + \mathbf{D}_p\mathbf{A}\mathbf{D}_p^{-1}\|_p$, which yields $\lambda_{\max}(\mathbf{A}) = \|s\mathbf{I} + \mathbf{D}_p\mathbf{A}\mathbf{D}_p^{-1}\|_p - s$. By taking $s = 1/h$, for $h \downarrow 0$, we obtain (9-CT).

Thus, from (8-CT) and (9-CT) we get the inequality

$$\mathscr{D}_t^+\mathscr{V}_p(\mathbf{x}(t)) \le \alpha(\mathbf{A})\mathscr{V}_p(\mathbf{x}(t)), \quad \forall t \in \mathbb{R}_+. \tag{10-CT}$$

Now, if (10-CT) holds for $\mathbf{x}(t) = \mathbf{x}(t;0,\mathbf{x}_0)$, let us consider the differential equation $\dot{\mathbf{y}}(t) \le \alpha(\mathbf{A})\mathbf{y}(t)$ with the initial condition $\mathbf{y}(0) = \mathscr{V}_p(\mathbf{x}(0)) = \mathscr{V}_p(\mathbf{x}_0)$. According to Theorem 4.2.11 in [3], we have

$$\begin{aligned}
\mathscr{V}_p(\mathbf{x}(t;0,\mathbf{x}_0)) &\le \mathbf{y}(t) = e^{\alpha(\mathbf{A})t}\mathbf{y}(0) \\
&= e^{\alpha(\mathbf{A})t}\mathscr{V}_p(\mathbf{x}_0), \\
&\qquad \forall \mathbf{x}_0 \in \mathbb{R}^n,\ \forall t \in \mathbb{R}_+,
\end{aligned} \tag{11-CT}$$

proving that system (1-CT) is (M,β)-stable, with $M = 1$ and $\beta = \alpha(\mathbf{A})$, in accordance with (3-CT). □

Remark 3. Theorem 2 reveals new connections between the *Perron-Frobenius eigenstructure* and the dynamics of asymptotically stable positive linear systems.

(i) Besides the information on the long-term behavior referring to the role of the *right eigenvector* in guiding (as asymptote) any trajectory $\mathbf{x}(t)$ for $t \to \infty$ (e.g., [16]), we show that the system dynamics exhibit set-invariance properties for sets defined by both *right and left eigenvectors*. Indeed, for any p, $1 \le p \le \infty$, the existence of the "ideal" transient behavior relative to the norm $\mathscr{V}_p(\mathbf{x})$ (proven by Theorem 2) is equivalent to the invariance of the exponentially contractive sets

$$\mathscr{X}_p^c(t) = \left\{\mathbf{x} \in \mathbb{R}^n \mid \left\|\mathbf{D}_p\mathbf{x}\right\|_p \le c(\alpha(\mathbf{A}))^t\right\}, \quad t \in \mathbb{Z}_+,\ c > 0, \tag{12-DT}$$

and, respectively,

$$\mathscr{X}_p^c(t) = \left\{\mathbf{x} \in \mathbb{R}^n \mid \left\|\mathbf{D}_p\mathbf{x}\right\|_p \le ce^{\alpha(\mathbf{A})t}\right\}, \quad t \in \mathbb{R}_+,\ c > 0, \tag{12-CT}$$

with respect to system (1-DT) (resp., (1-CT)). In particular, for an arbitrary (but fixed) time $t \in \mathbb{Z}_+$ (resp., $t \in \mathbb{R}_+$) and an arbitrary (but fixed) constant $c > 0$, the Minkowski functional of the constant set $\mathscr{X}_p^c(t)$ defined by (12-DT) (resp., (12-CT)) equals $\mathscr{V}_p(x)$ multiplied by a positive scalar. It is worth saying that the invariance of the sets of form (12-DT) (resp., (12-CT)) has already been mentioned by our previous works [17, 18]. The invariance analysis for sets of form (12-DT) (resp., (12-CT)) can be extended to positive interval systems, by using Corollary 3 and Theorem 4 in our paper [19], dealing with dynamics defined by interval matrices.

(ii) The existence of the invariant contractive sets of form (12-DT) (resp., (12-CT)) defined for all p, $1 \le p \le \infty$, allows a deeper insight into the dynamics of positive linear systems, for which the classical property presented in literature (e.g., [4]) is the invariance of the nonnegative orthant $\mathbb{R}_{+}^{n}$. In fact, all the nonnegative sets $\overline{\mathscr{X}}_{p}^{c}(t) = \mathscr{X}_{p}^{c}(t) \cap \mathbb{R}_{+}^{n}$, $t \in \mathbb{Z}_{+}$, $c > 0$, with $\mathscr{X}_{p}^{c}(t)$ defined by (12-DT) are invariant with respect to positive system (1-DT), and all the nonnegative sets $\overline{\mathscr{X}}_{p}^{c}(t) = \mathscr{X}_{p}^{c}(t) \cap \mathbb{R}_{+}^{n}$, $t \in \mathbb{R}_{+}$, $c > 0$, with $\mathscr{X}_{p}^{c}(t)$ defined by (12-CT) are invariant with respect to positive system (1-CT).

Remark 4. The proof of Theorem 2 also highlights the following dynamical properties of system (1-DT) (resp., (1-CT)) typical to the "ideal" transient behavior:

(i) For any p, $1 \leq p \leq \infty$, $\mathscr{V}_{p}(\mathbf{x})$ can serve for system (1-DT) (resp., (1-CT)), as a Lyapunov function (in a form called "norm Lyapunov function"), which decreases with the fastest possible rate, namely, $\alpha(\mathbf{A})$, along the trajectories of system (1-DT) (resp., (1-CT)).

(ii) Propositions 1 and 2 and Remark 1 in [9] refer to the existence of norm Lyapunov functions corresponding to $p \in \{1, 2, \infty\}$, but the possibility of ensuring the fastest decreasing rate $\alpha(\mathbf{A})$ (when the Perron-Frobenius eigenvectors are used) is not investigated.

(iii) The semigroup of operators $\mathbf{A}^{t}$, $t \in \mathbb{Z}_{+}$ (resp., $e^{\mathbf{A}t}$, $t \in \mathbb{R}_{+}$) is contractive relative to any operator norm subordinated to a vector norm of form $\mathscr{V}_{p}(\mathbf{x})$, $1 \leq p \leq \infty$. For all $1 \leq p \leq \infty$, the contraction rate is precisely $\alpha(\mathbf{A})$.

(iv) Form (7-DT//CT) of the scaling operator $\mathbf{D}_{p}$ uses the right and left eigenvectors of $\mathbf{A}$, uniquely defined by the norm equalities $\|\mathbf{v}\|_{\infty} = 1$, $\|\mathbf{w}\|_{\infty} = 1$. Obviously, Theorem 2 holds true if $\mathbf{D}_{p}$ is replaced by any diagonal matrix $k\mathbf{D}_{p}$, $k > 0$, fact showing that the essential information is offered by *the directions* of the right and left Perron-Frobenius eigenvectors (i.e., the fulfillment of the conditions $\|\mathbf{v}\|_{\infty} = 1$, $\|\mathbf{w}\|_{\infty} = 1$ is not compulsory).

Remark 5. Paper [13] addresses the transient behavior of positive linear systems and considers state-space transforms defined as in (7-DT//CT) by matrix $\mathbf{D}_{p}$. However, these transforms of type (7-DT//CT) are not regarded as individual scalings of the state-space variables (as illustrated by Theorem 2). The $\mathbf{D}_{p}$-weighted norms are used in the (M, β)-stability analysis in the sense of eccentricity with respect to the nonweighted norms, and, therefore, the approach misses the sharp interpretation of "ideal" transient behavior for the scaled variables. This is because the eccentricity caused by the $\mathbf{D}_{p}$-scaling can be produced by various nondiagonal weighting matrices, where the practical meaning of the state-space variables differs drastically from the original form to the weighted form. Moreover, all the results presented by the cited paper are limited to the particular cases $p \in \{1, 2, \infty\}$.

3. Results for Positive Systems Defined by Reducible Matrices

Throughout this section, matrix $\mathbf{A}$ that defines the dynamics of system (1-DT) (resp., (1-CT)) is *reducible* (i.e., the oriented graph $G(\mathbf{A})$ associated with $\mathbf{A}$ is not strongly connected). Matrix $\mathbf{A}$ is nonnegative (resp., essentially nonnegative). To deal with the reducibility of matrix $\mathbf{A}$, we consider two distinct cases, specified by the structure of the communication classes of matrix $\mathbf{A}$, for example, Section 3, Chapter 2 in [8].

3.1. All Communication Classes of Matrix **A** *Are Basic and Final.* Theorem 2 can still be used if all the communication classes are basic and final. Indeed Theorem 3.14, Chapter 2, from [8] guarantees the existence of positive right and left eigenvectors $\mathbf{v} \gg 0$, $\mathbf{w} \gg 0$, associated with the multiple eigenvalue $\lambda_{\max}(\mathbf{A})$. (In other words, we are able to generalize the "**A** irreducible" case, where the eigenvalue $\lambda_{\max}(\mathbf{A})$ is unique.) By a proof similar to the proof of the Theorem presented in [15], one can show that equality (9-DT) (resp., equality (9-CT)) holds true for **A** nonnegative (resp., essentially nonnegative).

3.2. Structure of Communication Classes of Matrix **A** *Is Different from Section 3.1.* There exists nonnegative right and left eigenvectors associated with the eigenvalue $\lambda_{\max}(\mathbf{A})$, but at least one of them contains 0 elements, in accordance with Theorems 3.10 and 3.14, Chapter 2, from [8]. Hence, the diagonal operator $\mathbf{D}_{p}$ defined by (7-DT//CT) cannot be used any longer.

However, the same type of information becomes available, if instead of **A** we consider the slightly modified matrix:

$$\mathbf{A}(\delta) = \mathbf{A} + \delta\mathbf{U}, \quad \delta > 0,$$
$$\mathbf{U} = \left[u_{ij}\right],$$
$$i, j = 1, \ldots, n, \; u_{ij} = \begin{cases} 1, & \text{if } a_{ij} = 0, \; i \neq j, \\ 0, & \text{otherwise}, \end{cases} \tag{13-DT//CT}$$

where δ is small enough. Matrix $\mathbf{A}(\delta)$ is irreducible, so that the p-type Perron-Frobenius scaling operator $\mathbf{D}_{p}(\delta)$ is well defined in accordance with (7-DT//CT) applied to $\mathbf{A}(\delta)$ as follows:

$$\mathbf{D}_{p}(\delta) = \begin{cases} \operatorname{diag}\{[(w_{1}(\delta)) \cdots (w_{n}(\delta))]\}, & \text{if } p = 1, \\ \operatorname{diag}\left\{\left[(w_{1}(\delta))^{1/p}(v_{1}(\delta))^{(-p+1)/p} \cdots (w_{n}(\delta))^{1/p}(v_{n}(\delta))^{(-p+1)/p}\right]\right\}, & \text{if } 1 < p < \infty, \\ \operatorname{diag}\left\{\left[(v_{1}(\delta))^{-1} \cdots (v_{n}(\delta))^{-1}\right]\right\}, & \text{if } p = \infty, \end{cases} \tag{14-DT//CT}$$

where $\mathbf{v}(\delta) \gg 0$, $\mathbf{w}(\delta) \gg 0$ are the Perron-Frobenius eigenvectors of $\mathbf{A}(\delta)$, which satisfy $\|\mathbf{v}(\delta)\|_\infty = 1$, $\|\mathbf{w}(\delta)\|_\infty = 1$.

Note that, for simplicity, in (13-DT//CT) one can use a matrix $\mathbf{U} = [u_{ij}]$ with $u_{ij} = 1$, $i, j = 1, \dots, n$, and the results of the current section preserve their validity.

Theorem 6. *Let $1 \le p \le \infty$. Consider an asymptotically stable, positive linear system of form* (1-DT) *(resp.,* (1-CT)*), with matrix $\mathbf{A}$ reducible. Let matrix $\mathbf{D}_p(\delta)$ be defined by* (14-DT//CT).

Then, for any $\varepsilon > 0$ arbitrarily small, there exists $\delta(\varepsilon) > 0$ such that, for each $\delta \in (0, \delta(\varepsilon)]$, system (1-DT) *(resp.,* (1-CT)*) is $(1, \alpha(\mathbf{A})+\varepsilon)$-stable relative to the norm $\mathscr{V}_p^\delta(x) = \|\mathbf{D}_p(\delta)\mathbf{x}\|_p$, where $\mathbf{D}_p(\delta)$ is the scaling operator defined by* (14-DT//CT).

Proof. For any $1 \le p \le \infty$, $\mathscr{V}_p^\delta(\mathbf{x})$ is a vector norm, defined by the standard p-norm, weighted by the scaling operator $\mathbf{D}_p(\delta)$ (14-DT//CT).

(a) In the DT case, for any $1 \le p \le \infty$, we first prove that

$$\begin{aligned}\lambda_{\max}(\mathbf{A}) &\le \left\|\mathbf{D}_p(\delta)\mathbf{A}\left(\mathbf{D}_p(\delta)\right)^{-1}\right\|_p \\ &\le \lambda_{\max}(\mathbf{A}(\delta)) < \lambda_{\max}(\mathbf{A}) + \varepsilon \\ &\quad \forall \varepsilon > 0,\ \exists \delta(\varepsilon) > 0 : \forall \delta \in (0, \delta(\varepsilon)]. \end{aligned} \tag{15-DT}$$

This proof relies on the following steps:

(i) The inequality $\lambda_{\max}(\mathbf{A}) \le \|\mathbf{D}_p(\delta)\mathbf{A}(\mathbf{D}_p(\delta))^{-1}\|_p$ is obvious.

(ii) For any $\mathbf{y} \in \mathbb{R}^n$, we can write the componentwise inequalities $|(\mathbf{D}_p(\delta)\mathbf{A}(\mathbf{D}_p(\delta))^{-1})\mathbf{y}| \le |(\mathbf{D}_p(\delta)\mathbf{A}(\mathbf{D}_p(\delta))^{-1})|\mathbf{y}|| \le |(\mathbf{D}_p(\delta)\mathbf{A}(\delta)(\mathbf{D}_p(\delta))^{-1})|\mathbf{y}||$ and the monotonicity of the vector p-norms implies $\|(\mathbf{D}_p(\delta)\mathbf{A}(\mathbf{D}_p(\delta))^{-1})\mathbf{y}\|_p \le \|(\mathbf{D}_p(\delta)\mathbf{A}(\mathbf{D}_p(\delta))^{-1}\|_p\|\mathbf{y}\|_p$.

(iii) The matrix norm means $\|\mathbf{D}_p(\delta)\mathbf{A}(\mathbf{D}_p(\delta))^{-1}\|_p = \max_{\|\mathbf{y}\|_p=1}\|(\mathbf{D}_p(\delta)\mathbf{A}(\mathbf{D}_p(\delta))^{-1})\mathbf{y}\|_p \le \|\mathbf{D}_p(\delta)\mathbf{A}(\delta)(\mathbf{D}_p(\delta))^{-1}\|_p$.

(iv) Equality (9-DT) written for $\mathbf{A}(\delta)$ and the diagonal operator $\mathbf{D}_p(\delta)$ ensures $\|\mathbf{D}_p(\delta)\mathbf{A}(\delta)(\mathbf{D}_p(\delta))^{-1}\|_p = \lambda_{\max}(\mathbf{A}(\delta))$.

(v) Since the Perron-Frobenius eigenvalue $\lambda_{\max}(\mathbf{A}(\delta)) = \lambda_{\max}(\mathbf{A} + \delta\mathbf{U})$ is continuous and strictly increasing with respect to $\delta > 0$ (Theorem 8.1.18 in [20])), we also have $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0 : \forall \delta \in (0, \delta(\varepsilon)]$, $\lambda_{\max}(\mathbf{A}(\delta)) < \lambda_{\max}(\mathbf{A}) + \varepsilon$.

Thus, the proof of (15-DT) is completed and we can use it for an arbitrary trajectory of system (1-DT). In accordance with (8-DT), where $\mathscr{V}_p^\delta(\mathbf{x}) = \|\mathbf{D}_p(\delta)\mathbf{x}\|_p$, we can write

$$\begin{aligned}\mathscr{V}_p^\delta(\mathbf{x}(t+1)) &\le \left\|\mathbf{D}_p(\delta)\mathbf{A}\left(\mathbf{D}_p(\delta)\right)^{-1}\right\|_p \mathscr{V}_p^\delta(\mathbf{x}(t)) \\ &< \left(\lambda_{\max}(\mathbf{A}) + \varepsilon\right)\mathscr{V}_p^\delta(\mathbf{x}(t)). \end{aligned} \tag{16-DT}$$

Since $\lambda_{\max}(\mathbf{A}) = \alpha(\mathbf{A})$, we get

$$\mathscr{V}_p^\delta(\mathbf{x}(t;0,\mathbf{x}_0)) < (\alpha(\mathbf{A}) + \varepsilon)^t \mathscr{V}_p^\delta(\mathbf{x}_0), \quad \forall \mathbf{x}_0 \in \mathbb{R}^n,\ \forall t \in \mathbb{Z}_+, \tag{17-DT}$$

proving that system (1-DT) is (M, β)-stable, with $M = 1$ and $\beta = \alpha(\mathbf{A}) + \varepsilon$, in accordance with (3-DT).

(b) In the CT case, for any $1 \le p \le \infty$, we first prove that

$$\begin{aligned}\lambda_{\max}(\mathbf{A}) &\le \lim_{h\downarrow 0}\frac{1}{h}\left[\left\|\mathbf{I} + h\mathbf{D}_p(\delta)\mathbf{A}\left(\mathbf{D}_p(\delta)\right)^{-1}\right\|_p - 1\right] \\ &\le \lambda_{\max}(\mathbf{A}(\delta)) < \lambda_{\max}(\mathbf{A}) + \varepsilon, \\ &\quad \forall \varepsilon > 0,\ \exists \delta(\varepsilon) > 0 : \forall \delta \in (0, \delta(\varepsilon)]. \end{aligned} \tag{15-CT}$$

This proof relies on the following steps:

(i) The inequality $\lambda_{\max}(\mathbf{A}) \le \lim_{h\downarrow 0}(1/h)[\|\mathbf{I} + h\mathbf{D}_p(\delta)\mathbf{A}(\mathbf{D}_p(\delta))^{-1}\|_p - 1]$ is ensured by Fact 11.15.7 in [1].

(ii) For small $h > 0$ we can write $0 \le \mathbf{I} + h\mathbf{D}_p(\delta)\mathbf{A}(\mathbf{D}_p(\delta))^{-1} \le \mathbf{I} + h\mathbf{D}_p(\delta)\mathbf{A}(\delta)(\mathbf{D}_p(\delta))^{-1}$ and, subsequently, $\|\mathbf{I} + h\mathbf{D}_p(\delta)\mathbf{A}(\mathbf{D}_p(\delta))^{-1}\|_p \le \|\mathbf{I} + h\mathbf{D}_p(\delta)\mathbf{A}(\delta)(\mathbf{D}_p(\delta))^{-1}\|_p$ that yields $\lim_{h\downarrow 0}(1/h)[\|\mathbf{I} + h\mathbf{D}_p(\delta)\mathbf{A}(\mathbf{D}_p(\delta))^{-1}\|_p - 1] \le \lim_{h\downarrow 0}(1/h)[\|\mathbf{I} + h\mathbf{D}_p(\delta)\mathbf{A}(\delta)(\mathbf{D}_p(\delta))^{-1}\|_p - 1]$.

(iii) The second equality in (9-CT) written for $\mathbf{A}(\delta)$ and the diagonal operator $\mathbf{D}_p(\delta)$ ensures $\lim_{h\downarrow 0}(1/h)[\|\mathbf{I} + h\mathbf{D}_p(\delta)\mathbf{A}(\delta)(\mathbf{D}_p(\delta))^{-1}\|_p - 1] = \lambda_{\max}(\mathbf{A}(\delta))$.

(iv) To complete the proof of (15-CT), we use the equalities $\lambda_{\max}(s\mathbf{I} + \mathbf{A}) = s + \lambda_{\max}(\mathbf{A})$, $\lambda_{\max}(s\mathbf{I} + \mathbf{A}(\delta)) = s + \lambda_{\max}(\mathbf{A}(\delta))$ and the same approach as in the proof of part (a).

For an arbitrary trajectory of system (1-CT), in accordance with (8-CT) where $\mathscr{V}_p^\delta(\mathbf{x}) = \|\mathbf{D}_p(\delta)\mathbf{x}\|_p$, we can write

$$\begin{aligned}\mathscr{D}_t^+\mathscr{V}_p^\delta(\mathbf{x}(t)) &\le \left(\mathscr{D}_t^+\left\|e^{\mathbf{D}_p(\delta)\mathbf{A}(\mathbf{D}_p(\delta))^{-1}t}\right\|_p\Big|_{t=0}\right)\mathscr{V}_p^\delta(\mathbf{x}(t)) \\ &< \left(\lambda_{\max}(\mathbf{A}) + \varepsilon\right)\mathscr{V}_p^\delta(\mathbf{x}(t)), \quad \forall t \in \mathbb{R}_+, \end{aligned} \tag{16-CT}$$

since $\mathscr{D}_t^+\|e^{\mathbf{D}_p(\delta)\mathbf{A}(\mathbf{D}_p(\delta))^{-1}t}\|_p|_{t=0} = \lim_{h\downarrow 0}(1/h)[\|\mathbf{I} + h\mathbf{D}_p(\delta)\mathbf{A}(\mathbf{D}_p(\delta))^{-1})\|_p - 1]$ as per Fact 11.15.7 in [1].

By using the same proof as for implication (10-CT) $\Rightarrow$ (11-CT), as well as the equality $\lambda_{\max}(\mathbf{A}) = \alpha(\mathbf{A})$, we get

$$\mathscr{V}_p^\delta(\mathbf{x}(t;0,\mathbf{x}_0)) < e^{(\alpha(\mathbf{A})+\varepsilon)t}\mathscr{V}_p^\delta(\mathbf{x}_0), \quad \forall \mathbf{x}_0 \in \mathbb{R}^n,\ \forall t \in \mathbb{R}_+, \tag{17-CT}$$

proving that system (1-CT) is (M, β)-stable, with $M = 1$ and $\beta = \alpha(\mathbf{A}) + \varepsilon$, in accordance with (3-CT). □

Remark 7. If system (1-DT) (resp., (1-CT)) is asymptotically stable, for the concrete use of Theorem 6, we chose $\varepsilon > 0$ such that $\lambda_{\max}(\mathbf{A}) + \varepsilon < 1$ (resp., $\lambda_{\max}(\mathbf{A}) + \varepsilon < 0$) and then we search for a $\delta^* > 0$ such that $\lambda_{\max}(\mathbf{A}) < \lambda_{\max}(\mathbf{A}(\delta^*)) < \lambda_{\max}(\mathbf{A})+\varepsilon$, for both DT and CT case. This search requires the computation of $\lambda_{\max}(\mathbf{A}(\delta))$ for some $\delta > 0$ and relies on the fact that $\lambda_{\max}(\mathbf{A}(\delta))$ is continuous and strictly increasing with respect to $\delta > 0$ (as mentioned in the proof of Theorem 6). Once such $\delta^* > 0$ is found, the diagonal operator $\mathbf{D}_p(\delta^*)$ is built in accordance with (14-DT//CT).

Remark 8. If system (1-DT) (resp., (1-CT)) is asymptotically stable and matrix $\mathbf{A}$ has the structure of the communication classes considered by the current subsection, Theorem 2 can be used in the following particular cases:

(i) For $p = 1$, if the basic and final classes of $\mathbf{A}^T$ coincide, then the diagonal operator $\mathbf{D}_1$ can be built in accordance with (7-DT//CT), since there exists a positive left eigenvector $\mathbf{w} \gg 0$ (as per Theorem 3.10, Chapter 2, from [8]).

(ii) For $p = \infty$, if the basic and final classes of $\mathbf{A}$ coincide, then the diagonal operator $\mathbf{D}_\infty$ can be built in accordance with (7-DT//CT), since there exists a positive right eigenvector $\mathbf{v} \gg 0$ (as per Theorem 3.10, Chapter 2, from [8]).

4. Case Studies

Example 1. Consider the electric circuit in Figure 1 that was also used in [21], where the following state-space model is given:

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \mathbf{A} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \mathbf{B} e(t),$$

$$\mathbf{A} = \frac{1}{R_1(R_2+R_3)+R_2R_3} \cdot \begin{bmatrix} \dfrac{-(R_2+R_3)}{C_1} & \dfrac{R_3}{C_1} \\ \dfrac{R_3}{C_2} & \dfrac{-(R_1+R_3)}{C_2} \end{bmatrix}, \qquad (18\text{-CT})$$

$$\mathbf{B} = \frac{1}{R_1(R_2+R_3)+R_2R_3} \begin{bmatrix} \dfrac{R_2}{C_1} \\ \dfrac{R_1}{C_2} \end{bmatrix}.$$

For $e(t) \equiv 0$ and $R_1 = 10^6\,\Omega$, $R_2 = 1.5 \cdot 10^6\,\Omega$, $R_3 = 2 \cdot 10^6\,\Omega$, $C_1 = 3 \cdot 10^{-8}$ F, $C_2 = 10^{-8}$ F, the circuit dynamics are described by

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \mathbf{A} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} -17.9487 & 10.2564 \\ 30.7692 & -46.1538 \end{bmatrix}. \qquad (19\text{-CT})$$

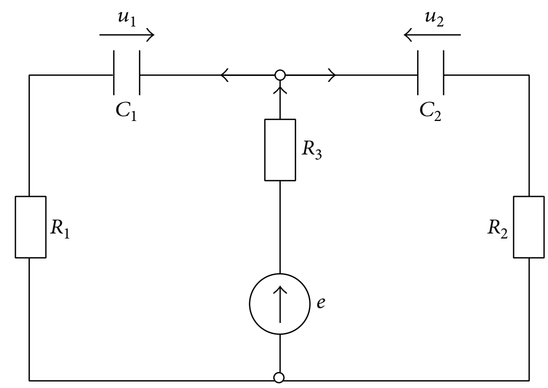


FIGURE 1: Electrical circuit used in Example 1.

Matrix $\mathbf{A}$ is irreducible. For the numerical values of the entries presented above, it has the Perron-Frobenius eigenvalue $\alpha(\mathbf{A}) = -9.3695$, with the associated right eigenvector $\mathbf{v} = [1.0000\ \ 0.8365]^T$ and left eigenvector $\mathbf{w} = [1.0000\ \ 0.2788]^T$.

For the vector p-norms defined by $p \in \{2, \infty\}$, this example is able to offer nice illustrations of the differences between the concepts of $(1, \alpha(\mathbf{A}))$-stability and $(1, \alpha_{\|\,\|}(\mathbf{A}))$-stability.

Thus, the short-term behavior fulfills the condition of $(1, -7.1583)$-stability relative to the vector norm $\|\cdot\|_2$, meaning that

$$\left\| \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \right\|_2 \le \left\| \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} \right\|_2 e^{\alpha_{\|\,\|_2}(\mathbf{A})t} = \left\| \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} \right\|_2 e^{-7.1583t}, \qquad (20\text{-CT})$$

whereas the condition of $(1, -9.3695)$-stability is fulfilled relative to the vector norm $\mathscr{V}_2(\cdot) = \|\mathbf{D}_2 \cdot\|_2$, $\mathbf{D}_2 = \operatorname{diag}\{\sqrt{w_1}/\sqrt{v_1}, \sqrt{w_2}/\sqrt{v_2}\} = \operatorname{diag}\{1, 0.5773\}$, meaning that

$$\left\| \begin{bmatrix} u_1(t) \\ 0.5773u_2(t) \end{bmatrix} \right\|_2 \le \left\| \begin{bmatrix} u_1(0) \\ 0.5773u_2(0) \end{bmatrix} \right\|_2 e^{\alpha(\mathbf{A})t} = \left\| \begin{bmatrix} u_1(0) \\ 0.5773u_2(0) \end{bmatrix} \right\|_2 e^{-9.3695t}. \qquad (21\text{-CT})$$

Similarly, the short-term behavior fulfills the condition of $(1, -7.6923)$-stability relative to the vector norm $\|\cdot\|_\infty$, meaning that

$$\left\| \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \right\|_\infty \le \left\| \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} \right\|_\infty e^{\alpha_{\|\,\|_\infty}(\mathbf{A})t} = \left\| \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} \right\|_\infty e^{-7.6923t}, \qquad (22\text{-CT})$$

whereas the condition of $(1, -9.3695)$-stability is fulfilled relative to the vector norm $\mathscr{V}_{\infty}(\cdot) = \|\mathbf{D}_{\infty} \cdot \|_{\infty}$, $\mathbf{D}_{\infty} = \text{diag}\{1/v_1, 1/v_2\} = \text{diag}\{1, 1.1955\}$, meaning that

$$\begin{aligned}\left\| \begin{bmatrix} u_1(t) \\ 1.1955 u_2(t) \end{bmatrix} \right\|_{\infty} &\leq \left\| \begin{bmatrix} u_1(0) \\ 1.1955 u_2(0) \end{bmatrix} \right\|_{\infty} e^{\alpha(\mathbf{A})t} \\ &= \left\| \begin{bmatrix} u_1(0) \\ 1.1955 u_2(0) \end{bmatrix} \right\|_{\infty} e^{-9.3695t}. \end{aligned} \quad \text{(23-CT)}$$

Both cases discussed above emphasize the role played by the scaling operator $\mathbf{D}_p$ defined by (7-DT//CT) for $p \in \{2, \infty\}$, when we are interested in describing the short-term behavior with the best decreasing rate (i.e., $\alpha(\mathbf{A}) = -9.3695$).

Note that the decreasing rate $\alpha(\mathbf{A}) = -9.3695$ can also be obtained by using $\| \cdot \|_1$ and the scaling operator $\mathbf{D}_1 = \text{diag}\{[w_1, w_2]\} = \text{diag}\{[1.0000, 0.2788]\}$, meaning that

$$\begin{aligned}\left\| \begin{bmatrix} u_1(t) \\ 0.2788 u_2(t) \end{bmatrix} \right\|_{1} &\leq \left\| \begin{bmatrix} u_1(0) \\ 0.2788 u_2(0) \end{bmatrix} \right\|_{1} e^{\alpha(\mathbf{A})t} \\ &= \left\| \begin{bmatrix} u_1(0) \\ 0.2788 u_2(0) \end{bmatrix} \right\|_{1} e^{-9.3695t}. \end{aligned} \quad \text{(24-CT)}$$

In other words, the condition of $(1, \alpha(\mathbf{A}))$-stability is also fulfilled relative to the vector norm $\mathscr{V}_1(\cdot) = \|D_1 \cdot \|_1$, but we cannot talk about $(1, \alpha_{\|\,\|_1}(\mathbf{A}))$-stability relative to the vector norm $\| \cdot \|_1$, since $\alpha_{\|\,\|_1}(\mathbf{A}) = 12.8205 > 0$.

Example 2. Consider the three-compartment mammillary CPB (cardiopulmonary bypass) model used in pharmacokinetics and discussed in Subsection 4.3 of [6]. The model has form (1-CT) with

$$\mathbf{A} = \begin{bmatrix} -(k_{11} + k_{21} + k_{31}) & k_{12} & k_{13} \\ k_{21} & -k_{12} & 0 \\ k_{31} & 0 & -k_{13} \end{bmatrix}, \quad \text{(25-CT)}$$

where $k_{11} = 0.001$, $k_{21} = 0.2$, $k_{12} = 0.2$, $k_{31} = 0.01$, $k_{13} = 0.02$ are the values used in the simulation example at page 122 in [6]. The mentioned example rules out partial monotonicity with respect to any compartment. Despite the lack of this property, our Theorem 2 is able to prove the existence of scaled p-norms relative to which the considered system has an "ideal" transient behavior; that is, it is $(1, \alpha(\mathbf{A}))$-stable.

Matrix $\mathbf{A}$ is irreducible. For the numerical values of the entries presented above, it has the Perron-Frobenius eigenvalue $\alpha(\mathbf{A}) = -0.0004$, with the associated right eigenvector $\mathbf{v} = [0.9980 \;\; 1.0000 \;\; 0.5091]^T$ and left eigenvector $\mathbf{w} = [0.9801 \;\; 0.9821 \;\; 1.0000]^T$. Thus, for any $1 \leq p \leq \infty$, Theorem 2 ensures the $(1, \alpha(\mathbf{A}))$-stability, relative to vector norm $\mathscr{V}_p(\mathbf{x}) = \|\mathbf{D}_p \mathbf{x}\|_p$, where $\mathbf{D}_p$ is the scaling operator defined by (7-DT//CT).

Our example provides graphical plots that support the intuitive understanding of the theoretical result stated by Theorem 2, for the frequently used vector p-norms. Thus, for each $p \in \{1, 2, \infty\}$, we simulate the system evolution for two initial conditions; namely, $\mathbf{x}_0 = [0.5 \;\; 1 \;\; 1]^T$ (studied at page 122 in [6]) and $\tilde{\mathbf{x}}_{0(p)} \neq \mathbf{x}_0$ with $\mathscr{V}_p(\tilde{\mathbf{x}}_{0(p)}) = \mathscr{V}_p(\mathbf{x}_0)$, and the simulation results are given in two distinct figures. Each figure displays the time dependence of the following functions:

(i) the three state variables plotted in *black* (by using line-types similar to Figure 4.2 in [6], i.e., solid line for $x_1(t)$, dashdot line for $x_2(t)$, dashed line for $x_3(t)$),

(ii) the left-hand side of inequality (3-CT) in Definition 1 plotted in *red*,

(iii) the right-hand side of inequality (3-CT) in Definition 1 plotted in *blue*.

These figures offer a nice graphical illustration for the fulfillment of the $(1, \alpha(\mathbf{A}))$-stability condition, expressed by Definition 1 and tested by Theorem 2. For all plots, the simulation horizon was selected $t \in [0, 300]$, as corresponding to a characterization of the short-term dynamics and also permitting direct comparisons to the simulation results presented in Figure 4.2 in [6].

The results corresponding to the use of Theorem 2 with $p = 1$ are presented in Figure 2. Figure 2(a) considers the dynamics started from the initial condition $\mathbf{x}_0 = [0.5 \;\; 1 \;\; 1]^T$, and Figure 2(b) refers to dynamics started from the initial condition $\tilde{\mathbf{x}}_{0(1)} = [0.5 \;\; 0.5 \;\; 1.4910]^T$, for which we have the equality $\mathscr{V}_1(\tilde{\mathbf{x}}_{0(1)}) = \mathscr{V}_1(\mathbf{x}_0) = 2.4721$, where $\mathscr{V}_1(\mathbf{x}) = \|\mathbf{D}_1 \mathbf{x}\|_1$, $\mathbf{D}_1 = \text{diag}\{0.9801, 0.9821, 1.0000\}$. In both figures, the blue line is identical and depicts the exponentially decreasing function $\mathscr{V}_1(\tilde{\mathbf{x}}_{0(1)}) e^{\alpha(\mathbf{A})t} = \mathscr{V}_1(\mathbf{x}_0) e^{\alpha(\mathbf{A})t}$ meaning the right-hand side of inequality (3-CT) in Definition 1. The red line depicts the function $\mathscr{V}_1(\mathbf{x}(t; 0, \mathbf{x}_0))$ in Figure 2(a) and the function $\mathscr{V}_1(\mathbf{x}(t; 0, \tilde{\mathbf{x}}_{0(1)}))$ in Figure 2(b), meaning the left-hand side of inequality (3-CT) in Definition 1 for the two discussed cases. Figures 2(a) and 2(b) graphically illustrate the inequality $\mathscr{V}_1(\mathbf{x}(t; 0, \mathbf{x}_0)) \leq \mathscr{V}_1(\mathbf{x}_0) e^{\alpha(\mathbf{A})t}$ and, respectively, the inequality $\mathscr{V}_1(\mathbf{x}(t; 0, \tilde{\mathbf{x}}_{0(1)})) \leq \mathscr{V}_1(\tilde{\mathbf{x}}_{0(1)}) e^{\alpha(\mathbf{A})t}$, for $t \in [0, 300]$, which, for this numerical example, are satisfied as equalities (and, consequently, the plots coincide for the left-hand sides (in red) and right-hand sides (in blue)).

The results corresponding to the use of Theorem 2 with $p = 2$ are presented in Figure 3. Figure 3(a) considers the dynamics started from the initial condition $\mathbf{x}_0 = [0.5 \;\; 1 \;\; 1]^T$, and Figure 3(b) refers to dynamics started from the initial condition $\tilde{\mathbf{x}}_{0(2)} = [0.5 \;\; 0.5 \;\; 1.1726]^T$, for which we have the equality $\mathscr{V}_2(\tilde{\mathbf{x}}_{0(2)}) = \mathscr{V}_2(\mathbf{x}_0) = 1.7865$, where $\mathscr{V}_2(\mathbf{x}) = \|\mathbf{D}_2 \mathbf{x}\|_2$, $\mathbf{D}_2 = \text{diag}\{0.9910, 0.9910, 1.4015\}$. In both figures, the blue line is identical and depicts the exponentially decreasing function $\mathscr{V}_2(\tilde{\mathbf{x}}_{0(2)}) e^{\alpha(\mathbf{A})t} = \mathscr{V}_2(\mathbf{x}_0) e^{\alpha(\mathbf{A})t}$ meaning the right-hand side of inequality (3-CT) in Definition 1. The red line depicts the function $\mathscr{V}_2(\mathbf{x}(t; 0, \mathbf{x}_0))$ in Figure 3(a), and the function $\mathscr{V}_2(\mathbf{x}(t; 0, \tilde{\mathbf{x}}_{0(2)}))$ in Figure 3(b), meaning the left-hand side of inequality (3-CT) in Definition 1 for the two discussed cases. Figures 3(a) and 3(b) graphically illustrate the inequality $\mathscr{V}_2(\mathbf{x}(t; 0, \mathbf{x}_0)) \leq \mathscr{V}_2(\mathbf{x}_0) e^{\alpha(\mathbf{A})t}$, and, respectively, the inequality $\mathscr{V}_2(\mathbf{x}(t; 0, \tilde{\mathbf{x}}_{0(2)})) \leq \mathscr{V}_2(\tilde{\mathbf{x}}_{0(2)}) e^{\alpha(\mathbf{A})t}$, for $t \in [0, 300]$.

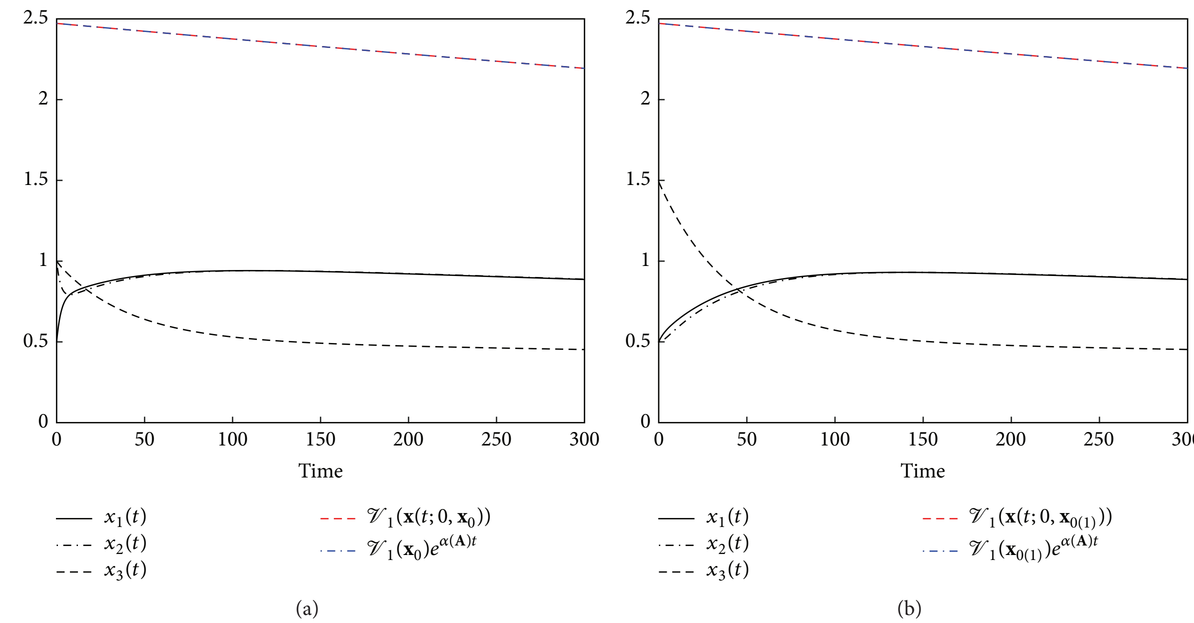


FIGURE 2: Results provided by Theorem 2 for Example 2 with $p = 1$ and initial condition (a) $\mathbf{x}_0 = [0.5 \ \ 1 \ \ 1]^T$ and (b) $\tilde{\mathbf{x}}_{0(1)} = [0.5 \ \ 0.5 \ \ 1.4910]^T$.

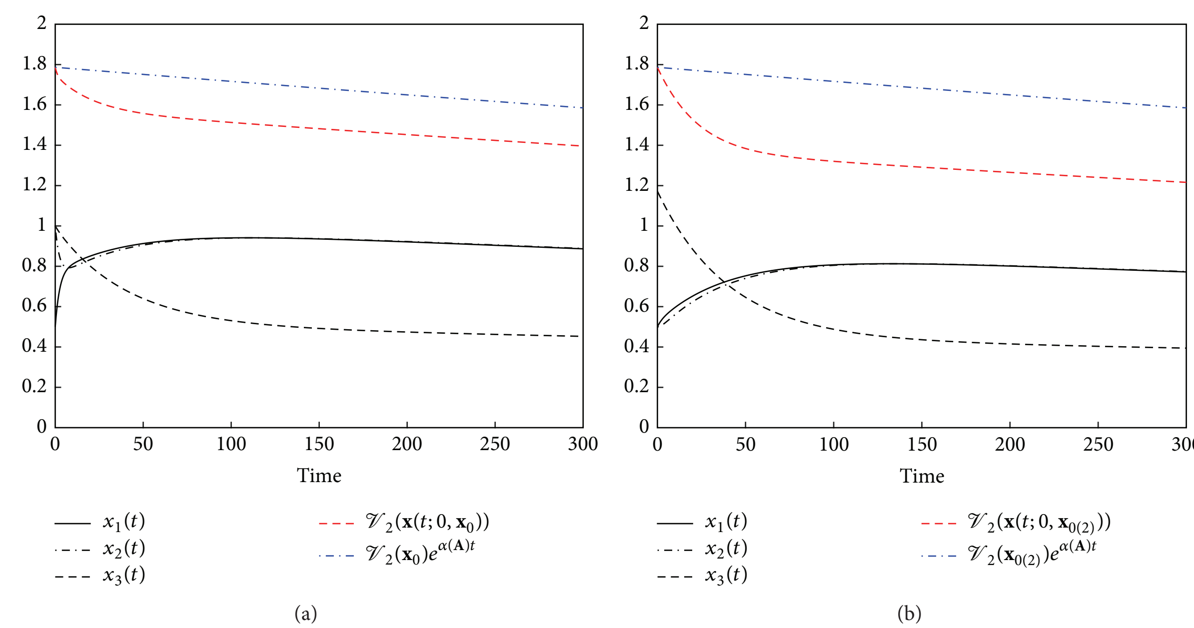


FIGURE 3: Results provided by Theorem 2 for Example 2 with $p = 2$ and initial condition (a) $\mathbf{x}_0 = [0.5 \ \ 1 \ \ 1]^T$ and (b) $\tilde{\mathbf{x}}_{0(2)} = [0.5 \ \ 0.5 \ \ 1.1726]^T$.

The results corresponding to the use of Theorem 2 with $p = \infty$ are presented in Figure 4. Figure 4(a) considers the dynamics started from the initial condition $\mathbf{x}_0 = [0.5 \ \ 1 \ \ 1]^T$, and Figure 4(b) refers to dynamics started from the initial condition $\tilde{\mathbf{x}}_{0(\infty)} = [1.5 \ \ 1 \ \ 1]^T$, for which we have the equality $\mathscr{V}_\infty(\tilde{\mathbf{x}}_{0(\infty)}) = \mathscr{V}_\infty(\mathbf{x}_0) = 1.9641$, where $\mathscr{V}_\infty(\mathbf{x}) = \|\mathbf{D}_\infty \mathbf{x}\|_\infty$, $\mathbf{D}_\infty = \mathrm{diag}\{1.0020, 1.0000, 1.9641\}$. In both figures, the blue line is identical and depicts the exponentially decreasing function $\mathscr{V}_\infty(\tilde{\mathbf{x}}_{0(\infty)})e^{\alpha(\mathbf{A})t} = \mathscr{V}_\infty(\mathbf{x}_0)e^{\alpha(\mathbf{A})t}$ meaning the right-hand side of inequality (3-CT) in Definition 1. The red line depicts the function $\mathscr{V}_\infty(\mathbf{x}(t;0,\mathbf{x}_0))$ in Figure 4(a), and the function $\mathscr{V}_\infty(\mathbf{x}(t;0,\tilde{\mathbf{x}}_{0(\infty)}))$ in Figure 4(b), meaning the left-hand side of inequality (3-CT) in Definition 1 for the two discussed cases. Figures 4(a) and 4(b) graphically illustrate

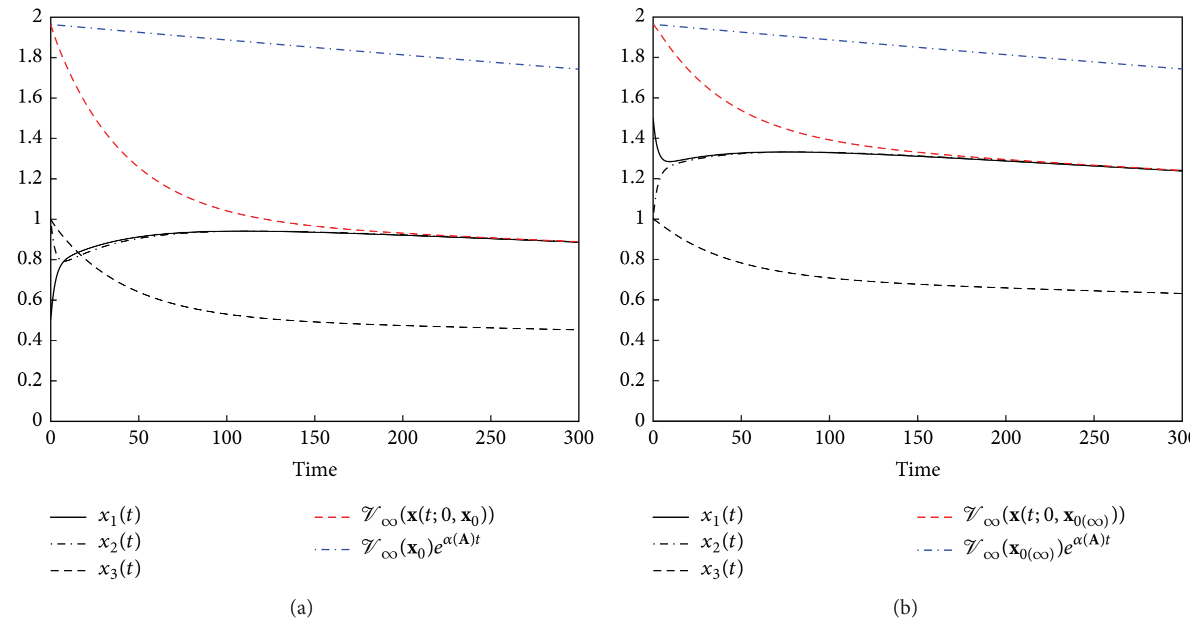


FIGURE 4: Results provided by Theorem 2 for Example 2 with $p = \infty$ and initial condition (a) $\mathbf{x}_0 = [0.5\ \ 1\ \ 1]^T$ and (b) $\tilde{\mathbf{x}}_{0(\infty)} = [1.5\ \ 1\ \ 1]^T$.

the inequality $\mathscr{V}_\infty(\mathbf{x}(t;0,\mathbf{x}_0)) \le \mathscr{V}_\infty(\mathbf{x}_0)e^{\alpha(\mathbf{A})t}$, and, respectively, the inequality $\mathscr{V}_\infty(\mathbf{x}(t;0,\tilde{\mathbf{x}}_{0(\infty)})) \le \mathscr{V}_2(\tilde{\mathbf{x}}_{0(\infty)})e^{\alpha(\mathbf{A})t}$, for $t \in [0,300]$.

5. Conclusions

The paper proves the existence of important connections between the Perron-Frobenius eigenstructure of a positive linear system and the short-term evolution of its state-space trajectories. These connections are explored by the help of the concept of (M,β)-stability relative to scaled vector p-norms, $1 \le p \le \infty$. If the time evolution of the trajectories is monitored by such a norm, then there exists a scaling operator built from the right and left Perron-Frobenius eigenvectors, which ensures an "ideal" transient behavior, meaning the least transient bound (i.e., $M = 1$), as well as the fastest exponential rate (i.e., β as close to $\alpha(\mathbf{A})$ as we want). The concrete expression of the scaling operator depends on $1 \le p \le \infty$, in the sense that for $p_1 \ne p_2$ the contributions of the Perron-Frobenius eigenvectors are different to the construction of the two scaling operators.

Our results cover both discrete- and continuous-time dynamics of positive linear systems. The analysis is organized so as to capture the differences between the cases where the system dynamics is defined by matrices irreducible and reducible, respectively. For the case of irreducible matrices we show that the fastest exponential rate in the "ideal" transient behavior means a unique value for β, namely, the fulfillment of the equality $\beta = \alpha(\mathbf{A})$.

We use our theoretical developments in two numerical case studies, both already discussed in literature by previous works. The first case study illustrates the differences between the concepts of $(1,\alpha(\mathbf{A}))$-stability and $(1,\alpha_{\|\,\|}(\mathbf{A}))$-stability associated with the operation of an electrical circuit. The second one constructs the scaling operators corresponding to $p \in \{1,2,\infty\}$ for a mammillary compartmental system and proves their role in ensuring the "ideal" short-term behavior.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors acknowledge the financial support of UEFISCDI Romania under Grant PN-II-ID-PCE-2011-3-1038.

References

[1] D. S. Bernstein, *Matrix Mathematics: Theory, Facts, and Formulas*, Princeton University Press, Princeton, NJ, USA, 2009.

[2] D. Hinrichsen and A. J. Pritchard, *Mathematical Systems Theory I, Modelling, State Space Analysis, Stability and Robustness*, Springer, Berlin, Germany, 2010.

[3] A. Michel and K. Wang, *Qualitative Theory of Dynamical Systems*, Marcel Dekker, New York, NY, USA, 1995.

[4] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*, John Wiley & Sons, New York, NY, USA, 2000.

[5] T. Kaczorek, *Positive 1D and 2D Systems*, Springer, London, UK, 2002.

[6] W. M. Haddad, V. S. Chellaboina, and Q. Hui, *Nonnegative and Compartmental Dynamical Systems*, Princeton University Press, Princeton, NJ, USA, 2010.

[7] A. Berman, M. Neumann, and R. J. Stern, *Nonnegative Matrices in Dynamic Systems*, Wiley-Interscience, 1989.

[8] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, New York, NY, USA, 1994.

[9] A. Rantzer, "Scalable control of positive systems," *European Journal of Control*, vol. 24, pp. 72–80, 2015.

[10] L. Benvenuti and L. Farina, "A tutorial on the positive realization problem," *IEEE Transactions on Automatic Control*, vol. 49, no. 5, pp. 651–664, 2004.

[11] S. U. Pillai, T. Suel, and S. Cha, "The Perron-Frobenius theorem," *IEEE Signal Processing Magazine*, vol. 22, no. 2, pp. 62–75, 2005.

[12] A. Berman, "Nonnegative matrices—old problems and new results," in *Positive Systems*, C. Commault and N. Marchands, Eds., vol. 341 of *Lecture Notes in Control and Information Sciences*, pp. 1–9, Springer, Berlin, Germany, 2006.

[13] D. Hinrichsen and E. Plischke, "Robust stability and transient behaviour of positive linear systems," *Vietnam Journal of Mathematics*, vol. 35, no. 4, pp. 429–462, 2007.

[14] E. Plischke and F. Wirth, "Duality results for the joint spectral radius and transient behavior," *Linear Algebra and Its Applications*, vol. 428, no. 10, pp. 2368–2384, 2008.

[15] J. Albrecht, "Minimal norms of nonnegative irreducible matrices," *Linear Algebra and Its Applications*, vol. 249, pp. 255–258, 1996.

[16] L. Farina, "Positive systems in the state space approach: main issues and recent results," in *Proceedings of the 15th International Symposium on Mathematical Theory of Networks and Systems (MTNS '02)*, 2002, https://www3.nd.edu/~mtns/papers/14900_1.pdf.

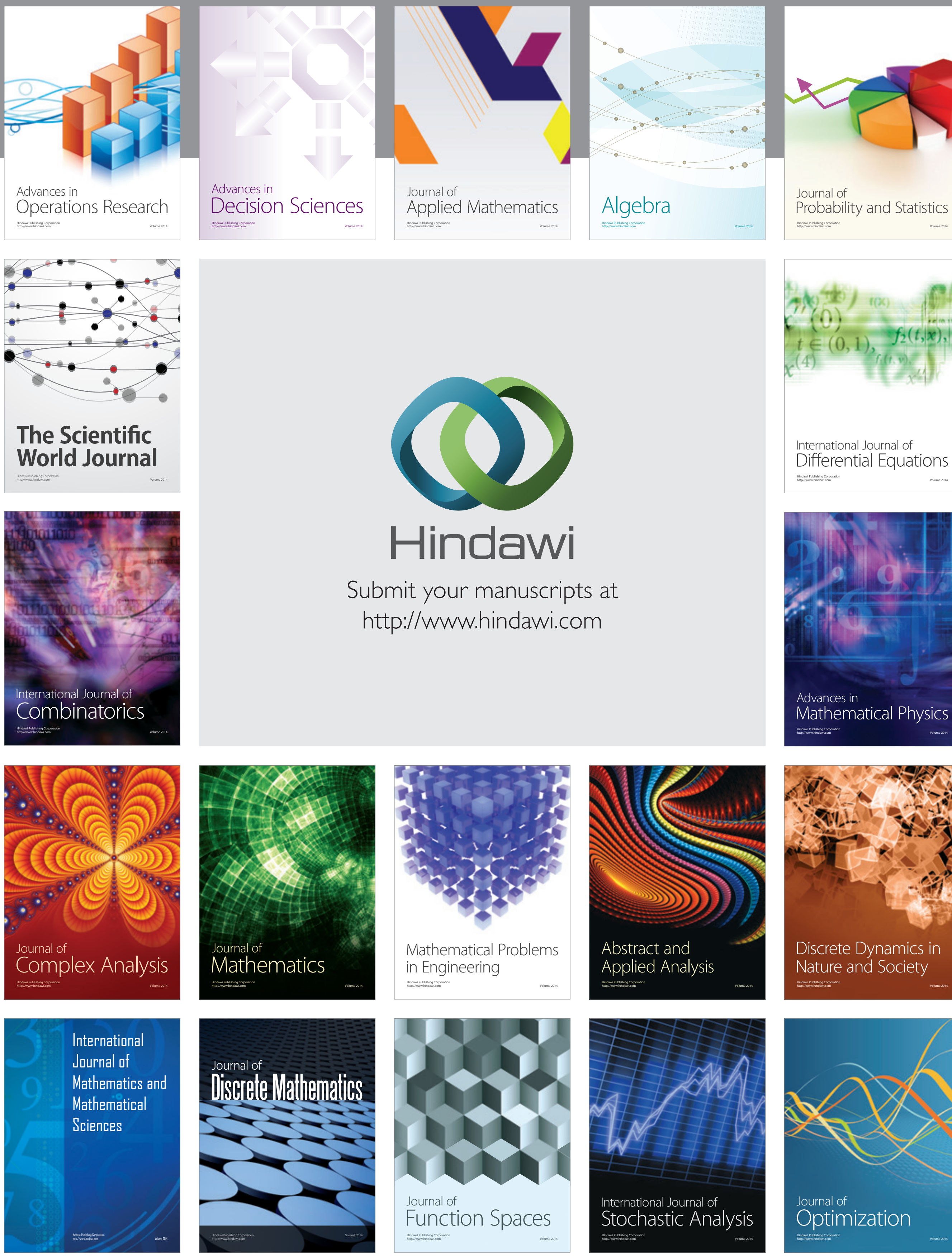
[17] O. Pastravanu and M. Voicu, "Generalized matrix diagonal stability and linear dynamical systems," *Linear Algebra and Its Applications*, vol. 419, no. 2-3, pp. 299–310, 2006.

[18] M.-H. Matcovschi and O. Pastravanu, "Perron-Frobenius theorem and invariant sets in linear systems dynamics," in *Proceedings of the 15th Mediterranean Conference on Control and Automation (MED '07)*, pp. 1–6, IEEE, Athens, Greece, July 2007.

[19] O. Pastravanu and M.-H. Matcovschi, "Diagonal stability of interval matrices and applications," *Linear Algebra and Its Applications*, vol. 433, no. 8–10, pp. 1646–1658, 2010.

[20] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1985.

[21] T. Kaczorek, "Strong stability of positive and compartmental linear systems," *Bulletin of the Polish Academy of Sciences: Technical Sciences*, vol. 56, no. 1, pp. 3–7, 2008.



Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Differential Equations
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization