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Positive observation problem for linear discrete positive systems

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Abstract— **This paper provides a new treatment of the theory of positive observers for discrete linear positive systems. The provided conditions for the existence of a positive observer are necessary and sufficient and solvable in terms of linear programming. Unlike the classical theory of observers, we also show that it is no longer possible to stabilize any unstable positive system by using positive observers. Numerical examples are given to show the applicability of the proposed approach.**

Key Words: positive systems, positive observers, positive observation, compartmental systems, linear programming.

I. Introduction

This paper considers a class of linear systems which have the property that its states are nonnegative whenever the initial conditions are nonnegative. In the literature, such systems are referred to be positive (see [4], [11], [7], [9] for general references). Most of the literature on positive linear systems focuses on the positive realization problem (see the tutorial paper [3]). Positive systems are connected to a wild spread of applications and their study is of continuing interest in control community. In practice, positive systems are very frequent, since many physical systems and chemical processes are by nature positive. For example, absolute temperatures, level of liquids in tanks and concentrations of chemicals are always nonnegative. The positivity constraint is inherent to other kind of systems such as in biology, chemistry, economics and sociology. Although the positivity of the system may depend on its nature, it may also result from physical limitations on its state.

This paper considers the *positive observation problem*, which consists of constructing positive observers, that is, observers that ensure the positivity of the estimated states. These observers are very important in practice, as non-negative values might be physically unrealistic (think of a negative length, level of liquid, etc). Positive observers can be used to derive upper and lower bounds on the observed states. That is, if the initial state of the observed system is unknown but bounded, the evolution of the real state will always be between the estimated states which are determined from any bounds on the real unknown initial states. Moreover, these estimated bounds are nonnegative and converge asymptotically to the observed nonnegative state.

As far as in our knowledge, little has been done in the literature on positive observation problem. The first study of this subject can be found in [12], where only a subclass of positive systems called compartmental systems is considered. We would like to point out that in [12] the gain of the positive observer was wrongly derived to be nonnegative. In fact, this is only necessary and sufficient for the single-output case with the output resulting from nonnegative linear transformations of the state. Unfortunately, for the multi-output case the positivity of the gain of the observer is only sufficient even if the output of the system results from non-negative linear transformations of the state. With regard to these previous works, this paper provides a new treatment for the positive observation problem of linear discrete positive systems. Moreover, we do not impose any sign restriction on the output of the system, that is, the linear output transformation of the state may not be necessary nonnegative. The proposed method is based on linear programming and follows the approach presented in [1]. We provide not only checkable necessary and sufficient conditions but also a simple approach to address numerically the determination of positive observers. Also, the stabilization problem by positive observer feedback is studied. In contrast to the classical theory of observers, we show that it is no longer possible to stabilize any unstable positive linear system by using positive observer feedback. The remainder of the paper is organized as follows. In section 2 some preliminary results are given. Section 3 treats and solves the positive observation problem in terms of linear programming. Section 4 shows the impossibility of the stabilization of any unstable positive system by using positive observers. Finally, section 5 gives some conclusions.

Notations: $\mathbf{R}_+^n$ denotes the non-negative orthant of the n-dimensional real space $\mathbf{R}^n$. M^T denotes the transpose of the real matrix M. For a real matrix M, $M > 0$ means that its components are positive: $M_{ij} > 0$, and $M \geq 0$ means that its components are nonnegative: $M_{ij} \geq 0$.

II. Preliminaries

Consider the following autonomous discrete linear system:

$$x_{k+1} \quad = \quad Ax_k, \tag{1}$$

with nonnegative initial conditions $x_0 \in \mathbf{R}_+^n$. Throughout the paper the following definitions will be used.

Definition 2.1: System (1) is said to be a *positive system* if the corresponding trajectory x_k is always nonnegative: $x_k \in \mathbf{R}_+^n$ for all integers k and nonnegative initial conditions.

Definition 2.2: A real matrix M is called a positive matrix if all its entries are nonnegative: $M_{ij} \geq 0$.

Definition 2.3: A real matrix M is called a Schur matrix if all its eigenvalues have magnitude less than one.

The following well-known and straightforward result relates positiveness of systems and their associated matrices.

Lemma 2.1: System (1) is positive if and only if A is a positive matrix: $A_{ij} \geq 0$.

In the following we provide necessary and sufficient conditions for the asymptotic stability of the autonomous system (1). Such conditions will be the basis of the main results of the paper.

Theorem 2.1: If System (1) is positive, then the following statements are equivalent:

- (i) System (1) is asymptotically stable for every initial condition $x_0 \in \mathbf{R}^n$ (i.e: A is a Schur matrix).
- (ii) System (1) is asymptotically stable for every nonnegative initial condition $x_0 \in \mathbf{R}_+^n$.
- (iii) System (1) is asymptotically stable for an arbitrary initial condition x_0 in the interior of $\mathbf{R}_+^n$.
- (iv) There exists $\lambda \in \mathbf{R}_+^n$ such that $(A - I)\lambda < 0$.

Proof: Expressing any initial condition as the difference of two nonnegative vectors, the equivalence between (i) and (ii) is straightforward. For the rest of the proof it suffices to prove that (iii) and iv are equivalent. Assume that (iii) holds for some initial condition x_0 in the interior of $\mathbf{R}_+^n$. Then by summing and arranging the terms of System 1 we obtain

$$x_k - x_0 = (A - I)\sum_{i=0}^{k-1} x_i < 0.$$

Since x_k goes to zero, then by taking the limit:

$$-x_0 = (A - I)\sum_{i=0}^{+\infty} x_i < 0.$$

Regarding to the fact that x_0 is non-negative we also have $\lambda = \sum_{i=0}^{+\infty} x_i > 0$, so it has been proved that (iii) implies (iv). Finally, to show that (iv) also implies (iii), we use the fact that the dual system $x_{k+1} = A^T x_k$ is positive and stable if and only if the original system (1) is positive and stable. Consequently, if condition (iv) is satisfied, the dual system is stable, since $v(x_k) = x_k^T \lambda$ is a Lyapunov function of the dual system, and the proof is complete. ■

III. Positive observers synthesis

This section presents necessary and sufficient conditions for the existence of observers of discrete positive systems, that give always positive observation of the trajectory, and tracts asymptotically the current state. It will be shown that this problem can be cast as an LP problem. Consider the following observed positive system:

$$\begin{array}{rcl} x_{k+1} & = & Ax_k, \\ y_k & = & Cx_k, \end{array} \tag{2}$$

for which the initial condition x_0 and the trajectory x_k are assumed to be nonnegative and unknown, but the output $y_k \in \mathbf{R}^r$ is known and not necessarily nonnegative (C may not be a positive matrix). The objective is to determine a nonnegative approximation $\hat{x}_k \geq 0$ of the state x_k, such that the error $\hat{x}_k - x_k$ converges asymptotically to zero. Following the classical approach of linear observers [10], an approximation $\hat{x}_k$ is given by a linear observer of the Luenberger form:

$$\hat{x}_{k+1} = (A - LC)\hat{x}_k + Ly_k, \tag{3}$$

where $L \in \mathbf{R}^{n\times r}$ is to be determined to ensure the positiveness of $\hat{x}_k$ and the vanishing of the error $\hat{x}_k - x_k$. The following result shows that additional conditions on the structure of the observer are needed.

Lemma 3.1: There exists a positive observer of system (2) if and only if $LC \geq 0$, $A - LC \geq 0$ and $A - LC$ is a Schur matrix.

Proof: The error associated to the observed state $e_k = \hat{x}_k - x_k$ must tend to zero. Since this error is also a solution to the linear system $e_{k+1} = (A - LC)e_k$, then $A - LC$ must be a Schur matrix.

Now, since the following augmented system must be positive:

$$\begin{pmatrix} x_{k+1} \\ \hat{x}_{k+1} \end{pmatrix} = \begin{bmatrix} A & 0 \\ LC & A - LC \end{bmatrix} \begin{pmatrix} x_k \\ \hat{x}_k \end{pmatrix},$$

then by using Lemma 2.1 we conclude that the conditions $LC \geq 0$ and $A - LC \geq 0$ are necessary and sufficient for the positiveness of the observer. ■

Remark 3.1: We stress out that in a previous result for positive observer [12] the gain of the observer was wrongly taken to be nonnegative $L \geq 0$. For the single-output system with positive matrix C this condition is also necessary and sufficient. Unfortunately, for the multi-output case this condition is only sufficient even if the matrix C is positive. The following result provides a computational approach for positive observers of system (2).

Theorem 3.1: The following statements are equivalent:

- (i) There exists a positive observer of System (2) of the form:

$$\hat{x}_{k+1} = (A - LC)\hat{x}_k + Ly_k$$

.

- (ii) There exists a matrix $L \in \mathbf{R}^{n\times r}$ such that $LC \geq 0$, $A - LC \geq 0$ and $A - LC$ is a Schur matrix.

- (iii) The following LP problem is feasible:

$$\begin{cases} (A^T - I)\lambda - C^T \sum_{i=1}^{n} z_i < 0, \\ \lambda > 0, \\ c_i^T z_j \geq 0 \text{ for } i, j = 1, \ldots, n, \\ a_{ji}\lambda_j - c_i^T z_j \geq 0, \end{cases} \quad (4)$$

where $A = [a_{ij}]$, $C = [c_1 \ \ldots \ c_n]$ and the variables are $\lambda = [\lambda_1 \ \ldots \ \lambda_n]^T \in \mathbf{R}^n$, $z_1, \ldots, z_n \in \mathbf{R}^r$. Moreover, a matrix L satisfying the statement (ii) can be calculated as

$$L^T = \left[\frac{1}{\lambda_1} z_1 \ \ldots \ \frac{1}{\lambda_n} z_n \right],$$

where the variables λ_i and z_i can be any feasible solution to the above LP problem.

Proof: The equivalence between (i) and (ii) is straightforward from Lemma 3.1.

Now it is shown that (ii) and (iii) are equivalent. First, note that $A - LC$ is a positive and Schur matrix if and only if its transpose is a positive and Schur matrix. Thus, it follows that the statement (ii) is equivalent to the existence of a matrix $L \in \mathbf{R}^{n \times r}$ satisfying the following conditions:

$$\begin{cases} C^T L^T \geq 0, \\ A^T - C^T L^T \text{is a positive matrix}, \\ A^T - C^T L^T \text{is a Schur matrix}. \end{cases}$$

Consequently it suffices to show that the statement (iii) is equivalent to the above conditions.

Now, assume that condition (iii) holds and define the matrix $L^T = [l_1, \ldots, l_n]$ with $l_i = \lambda_i^{-1} z_i$ for $i = 1, \ldots, n$. By calculation we have $L^T \lambda = \sum_{i=1}^{n} z_i$, which is then used in the first inequality of (4) to obtain $(A^T - I - C^T L^T)\lambda < 0$. Since $\lambda > 0$ then by using Theorem 2.1 we can conclude that $A^T - C^T L^T$ is a Schur matrix if it is a positive matrix. Now, it is easy to see that $A^T - C^T L^T$ is a positive matrix since from the last inequality of (4) we get

$$a_{ji} - c_i^T \frac{1}{\lambda_j} z_j = a_{ji} - c_i^T l_j = (A^T - C^T L^T)_{ij} \geq 0.$$

In the same manner $C^T L^T \geq 0$ can be obtained from the third inequality of (4).

The implication $(ii) \Rightarrow (iii)$ and the rest of the proof follows the same line of argument. ■

Remark 3.2: Since the error $e_k = \hat{x}_k - x_k$ is a solution to the linear system $e_{k+1} = (A - LC)e_k$ and $A - LC$ is a positive matrix, then the error satisfies a positive system. So that one can easily check that the error satisfies: $e_k \leq 0$ whenever $e_0 \leq 0$ (also by positivity $e_k \geq 0$ whenever $e_0 \geq 0$). As consequence, the designed observers can also be used to derive upper and lower bounds on the observed states. Effectively, if the initial state of the observed system is bounded: $0 \leq \bar{x}_1 \leq x_0 \leq \bar{x}_2$ the evolution of the real state x_k will always be between the estimated states $\hat{x}_{lower}$ and $\hat{x}_{upper}$:

$$\hat{x}_{lower} \leq x(0) \leq \hat{x}_{upper},$$

where $\hat{x}_{lower}$ has as initial condition $\bar{x}_1$ and $\hat{x}_{upper}$ has as initial condition $\bar{x}_2$ ($\hat{x}_{lower}$ and $\hat{x}_{upper}$ are constructed as in Theorem 3.1). Moreover, these estimated bounds are nonnegative and converge asymptotically to the observed state.

IV. Stabilization is not possible

In the preceding, we have shown how to solve the observation problem in terms of LP. In contrast with the classical theory of observers, we show here that the stabilization of positive systems by using positive observers is not possible. This fact can be also derived from a general analysis of monotone systems (see for example [6], [2]).

Consider the following controlled discrete linear system:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k, \end{aligned} \quad (5)$$

where the initial condition and the state are unknown and assumed to be nonnegative (i.e. $x_k \in \mathbf{R}_+^n$) under the input control signal $u_k \in \mathbf{R}^p$. The output $y_k \in \mathbf{R}^r$ is known and not necessarily nonnegative (C may be indefinite sign matrix). The aim here is to show that any positive observer ($\hat{x}_k \geq 0$) in the form

$$\hat{x}_{k+1} = (A - LC)\hat{x}_k + Bu_k + Ly_k, \quad (6)$$

which allow the observed state to be nonnegative $x_k \geq 0$, cannot stabilize asymptotically System (5), when an observer feedback law $u_k = K\hat{x}_k$ is utilized. To show this, we provide the following result.

Lemma 4.1: Assume that the matrix A is not Schur. Then, there does not exist a positive observer of the form (6) for system (5), such that the observer feedback law $u_k = K\hat{x}_k$ is asymptotically stabilizing.

Proof: Assume that A is not Schur and that there exist matrices $L \in \mathbf{R}^{n \times r}$ and $K \in \mathbf{R}^{p \times n}$ which fulfill the positiveness of x_k and $\hat{x}_k$, and their asymptotic convergence to zeros, under the feedback control $u_k = K\hat{x}_k$. We now show that such statement leads to a contradiction:

since the following augmented system must be positive:

$$\begin{pmatrix} x_{k+1} \\ \hat{x}_{k+1} \end{pmatrix} = \begin{bmatrix} A & BK \\ LC & A - LC + BK \end{bmatrix} \begin{pmatrix} x_k \\ \hat{x}_k \end{pmatrix}, \quad (7)$$

then by using Lemma 2.1 we have that the condition $BK \geq 0$ is necessary.

Now, by using the transformation:

$$T = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix},$$

it is easy to check that the dynamical matrix of the augmented system (7) is similar to the following matrix:

$$\begin{bmatrix} A - LC & 0 \\ LC & A + BK \end{bmatrix}.$$

Henceforth, if the augmented system (7) is asymptotically stable then necessarily $A + BK$ must be a Schur matrix, but this is impossible since we have shown that $BK \geq 0$. Thus, the fact that $A + BK$ is a Schur positive matrix leads to the contradiction that A is a Schur matrix. To see this, Theorem 2.1 implies the existence of $\lambda > 0$ such that $(A + BK - I)\lambda < 0$. So that as BK is positive, we necessarily have $(A - I)\lambda < 0$, which means, by using Theorem 2.1, that A is necessarily a Schur matrix and the proof is complete. ■

V. Numerical Example

Suppose we have the following positive system:

$$A = \begin{bmatrix} 0.7 & 0.4 \\ 0.1 & 0.5 \end{bmatrix}, \qquad C = \begin{bmatrix} -0.9 & -0.8 \end{bmatrix}$$

We want to design an observer that ensures that the estimated states are nonnegative and converge to the real value. Based on Theorem 3.1, the following inequalities must be fulfilled:

$$\begin{bmatrix} -0.3 & 0.1 & 0.9 & 0.9 \\ 0.4 & -0.5 & 0.8 & 0.8 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} < 0,$$

$$\begin{bmatrix} 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 0.9 \\ 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0.8 \\ 0 & -0.1 & 0 & -0.9 \\ -0.4 & 0 & -0.8 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} \leq 0,$$

since the gain of the observer is given by $L = \begin{bmatrix} \frac{z_1}{\lambda_1} \\ \frac{z_2}{\lambda_2} \end{bmatrix}$, one feasible solution to the above LP problem provides $L = \begin{bmatrix} -0.4381 \\ -0.0713 \end{bmatrix}$. As it has been previously pointed out in Remark 3.2 this observer can also be used to derive upper and lower bounds on the observed states. For example, if it is known that the initial conditions for the real system are in the range:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq x_0 \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then, if we denote x_{lower} the estimation of the state from the initial condition $\hat{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and x_{upper} the estimation of the state from the initial condition $\hat{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then the evolution of the real state x_k will always be between the estimated states x_{lower} and x_{upper}. Moreover, these estimated states are nonnegative and converge to the real value. These properties can be seen in Figures 1 and 2, that plot the state evolutions from random initial nonnegative conditions and random inputs u_k, with $B = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}$. It is possible to see that the observed states are nonnegative and converge to the real value, and correspond to lower and upper bounds on the real states.

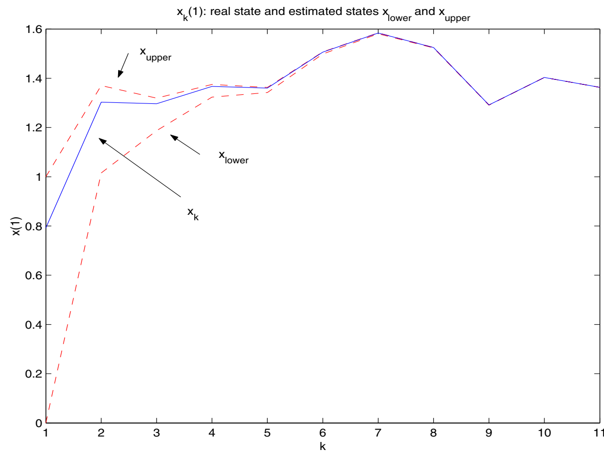

Fig. 1. Evolution of the first state and estimated bounds

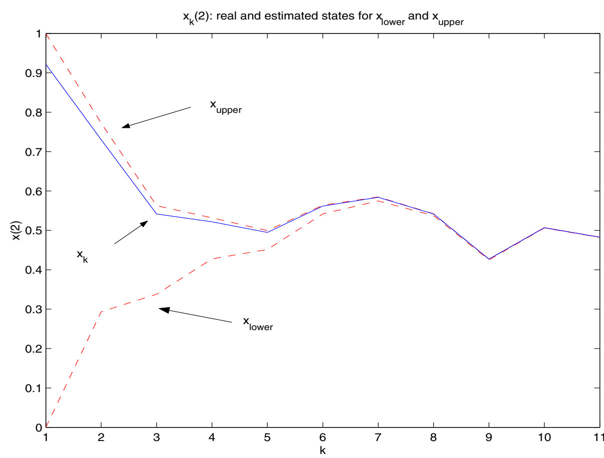

Fig. 2. Evolution of the second state and estimated bounds

VI. Conclusions

This paper have proposed a new approach to solve some synthesis problems for positive linear systems. The observation problem have been considered, and necessary and sufficient conditions for its solvability have been proposed. It has been shown that all the proposed conditions are solvable in terms of Linear Programming problems.

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