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Positive observation problem for linear time-delay positive systems

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Abstract

This paper deals with the problem of positive observation for linear time-delay systems for which the states take nonnegative values whenever the initial conditions are nonnegative. We focus on the design of positive observers (possibly with time-delay) which guarantee nonnegative estimates of the current states. We derive necessary and sufficient conditions for the existence of a positive observer (extended Luembergertype) and show that the solvability of the problem can be decided via standard linear programming techniques. Moreover, on the negative side, it is shown that one cannot stabilize any unstable positive time-delay system by using extended Luenberger type positive observers. In other words, the separation principle does not hold.

keywords: positive time-delay systems, positive observers, positive observation, compartmental systems, linear programming

1 Introduction

Differential delay systems known also as hereditary or systems with aftereffects, represent a class of infinite-dimensional systems which model propagation phenomena, population dynamics and many physical and chemical processes. As matter of fact, the reaction of real world systems to exogenous signals is never instantaneous and always infected by certain time delays. Such pathological phenomena can be adequately described by a mathematical model in which the behavior of the rate of the state is described by an equation including some information on the past evolution of the system. In general, for linear time-delay systems and independently of the representation type, the delay effects on the stability and control of dynamical systems (delays in the state and/or in the input) are problems of a great interest since the delay presence may induce complex behaviors (oscillations, instability, bad performances) for the closed-loop system. Also, small delays may destabilize some systems, however large delays may stabilize others. In contrast, for *positive* linear time-delay systems, the presence of delays does not affect the stability performance of the system [13, 13, 16, 9]. In particular, this paper shows that the convergence of the estimated state to actual state is insensitive to constant delays.

Despite the obvious relevance of positive time-delays systems in many systems engineering application areas, the theoretical foundations of such control systems are not sufficiently well understood and well developed in the *computational* sense. This paper focuses on one particularly relevant aspect of such systems, the *positive observation problem*, that is both theoretically challenging and practically relevant. Such problem consists of constructing positive observers, that is, observers that ensure the nonnegativity of the estimated states.

In the already existing literature, systems that are linear control systems and whose state variables take only nonnegative values are referred to be positive (see [6, 15, 8, 11] for general references). The aim of this paper is to present a new method and techniques for the analysis and the synthesis of linear positive observers for positive linear systems in presence of delays. The proposed approach is as simple as solving a Linear Programming (LP) problem. In fact, based on simple idea, this paper develops theoretical results with necessary and sufficient conditions, which turn out to be naturally translated into an LP problem.

Most of the literature on *linear positive systems* focuse on the positive realization problem (see the tutorial paper [5]). To the best of our knowledge, little has been done in the literature on positive observation problem. The first study of this subject can be found in [17], where only a subclass of positive systems called compartmental systems is considered. In [17] the gain of the positive observer was shown to be positive. In fact, this is only necessary and sufficient for the single-output case with the output resulting from a positive linear transformation of the state. Unfortunately, for the multi-output case the positivity of the gain of the observer is only sufficient even if the output of the system results from a positive linear transformation of the state. With regard to these previous works, this paper provides a new and complete treatment for the positive observation problem of positive linear time-delay systems. Moreover, we do not impose any sign restriction on the output of the system, that is, the linear output transformation of the state may not be necessary positive. The proposed method is based on linear programming and follows the approach presented in [4, 2, 3, 1]. We provide not only checkable necessary and sufficient conditions but also a simple approach to address numerically the determination of positive

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observers. We show that the designed observers can also be used to derive upper and lower bounds on the observed states. That is , if the initial state of the observed system is unknown but bounded, the evolution of the real state will always be between the estimated states. Moreover, these estimated bounds are positive and converge asymptotically to the observed state. Also, the stabilization problem by positive observer feedback is studied. In contrast to the classical theory of observers, we show that it is no longer possible to stabilize any unstable positive linear time-delays system by using positive observers as a feedback.

The remainder of the paper is organized as follows. In section 2 some preliminary results are given. Section 3 treats and solves the positive observation problem in terms of linear programming. Section 4 shows the impossibility of the stabilization of any unstable positive linear time-delay system by using positive observers. Finally, section 5 gives some conclusions.

Notations: $\Re^n_+$ denotes the non-negative orthant of the n-dimensional real space $\Re^n$. M^T denotes the transpose of the real matrix M. For a real matrix M, $M > 0$ means that its components are positive: $M_{ij} > 0$, and $M \geq 0$ means that its components are nonnegative: $M_{ij} \geq 0$. $\mathbf{diag}(\lambda)$ is the diagonal matrix whose diagonal is formed by the components of the vector λ.

2 Preliminaries

This section presents a precise setting for the positive observation problem of positive systems described by a differential delayed linear equation. Also, it provides some definitions and preliminary results that are used throughout the paper.

Consider the following observed system

$$\begin{aligned} \frac{dx}{dt} &= Ax + \sum_{i=1}^{m} A_i x(t-\tau_i), \\ y(t) &= Cx(t) + \sum_{i=1}^{m} C_i x(t-\tau_i) \in \Re^r, \end{aligned} \tag{1}$$

the given matrices $A, A_1, \ldots, A_m \in \Re^{n\times n}$ are time-invariant and $0 < \tau_1 < \cdots < \tau_m \equiv \tau$ are time delays associated with the system coordinates. The trajectory $x(\cdot)$ is assumed to be nonnegative and unknown, but its output $y(\cdot)$ is known and not necessarily nonnegative ($C, C_1 \ldots, C_m$ may be indefinite sign matrix).

Our objective is to determine a nonnegative approximation $\hat{x}(\cdot)$ of the state $x(\cdot)$, such that the error $e(t) \doteq \hat{x}(t) - x(t)$ converges asymptotically to zero. Based on the classical approach of linear observers [14], an extended Luenberguer-type observer $\hat{x}$ for System (1) is given by a linear dynamical delayed observer as follows:

$$\begin{aligned} \frac{d\hat{x}}{dt} &= A\hat{x} + \sum_{i=1}^{m} A_i \hat{x}(t-\tau_i) - L(\hat{y} - y), \\ \hat{y}(t) &= C\hat{x}(t) + \sum_{i=1}^{m} C_i \hat{x}(t-\tau_i), \end{aligned} \tag{2}$$

or equivalently,

$$\frac{d\hat{x}}{dt} = (A - LC)\hat{x} + \sum_{i=1}^{m} (A_i - LC_i)\hat{x}(t-\tau_i) + Ly, \tag{3}$$

where $L \in \Re^{n\times r}$ is the gain of the observer to be determined to fulfill the nonnegativity of $\hat{x}$ and its asymptotic convergence to the actual state.

Now, it can be easily shown that the estimated error associated to the observed state $e(t) = \hat{x}(t) - x(t)$ is governed by the following delayed system

$$\frac{d\hat{e}}{dt} = (A - LC)\hat{e} + \sum_{i=1}^{m} (A_i - LC_i)\hat{e}(t-\tau_i). \tag{4}$$

From here to the rest of this section, we only consider and study autonomous positive linear time-delay systems, in order to establish its properties and to analyze its stability. The obtained facts will be utilized further to derive our main result for the existence of positive observers. Thus, consider the following system:

$$\begin{aligned} \frac{dx}{dt} &= Ax + \sum_{i=1}^{m} A_i x(t-\tau_i), \\ x(t) &= \phi(t) \in \Re^n_+, \text{ for } -\tau \leq t \leq 0, \end{aligned} \tag{5}$$

Some definitions are stated for the rest of the paper.

Definition 2.1 *Given any nonnegative initial condition $\phi(t) \in \Re^n_+$ such that $x(t) = \phi(t)$ for $-\tau \leq t \leq 0$, System (5) is said to be positive if the corresponding trajectory is nonnegative: $x(t) \in \Re^n_+$ for all $t \geq 0$.*

Definition 2.2 *A real matrix M is called a* Metzler *matrix if its off-diagonal elements are nonnegative: $M_{ij} \geq 0$, $i \neq j$.*

Definition 2.3 *A real matrix M is called a* positive *matrix if all its elements are nonnegative: $M_{ij} \geq 0$.*

In fact, the positivity condition in the sense of Definition 2.1 and with regard to the dynamic of System (5) can be easily checked. In what follows, it is shown that one can determine whether a continuous-time delay system is positive or not by simply checking the sign of the entries of the matrices involved in the mathematical model of System (5). The following result, can be seen as an extension of a classical result in [15].

Lemma 2.1 *System (5) is positive if and only if A is a Metzler matrix and $A_1, \ldots, A_m$ are positive matrices.*

The following plays a key role and it shows that positive linear systems are necessarily monotone with regard to the initial conditions.

Lemma 2.2 *Assume that System (5) is positive. Let $\phi^1(\cdot)$ and $\phi^2(\cdot)$ be given initial conditions such that*

$$\phi^1(s) \leq \phi^2(s), \ \ \forall s : -\tau \leq s \leq 0,$$

also, consider their associated trajectories $x^1(\cdot)$ and $x^2(\cdot)$ solution to System (5). Then, we have

$$x^1(t) \leq x^2(t), \ \ \forall t \geq 0.$$

Proof: It suffices to choose $\phi(\cdot) \doteq \phi^2(\cdot) - \phi^1(\cdot) \geq 0$ as an initial condition and utilize the linearity and the positivity of System (5). $\diamondsuit$

Next, our aim is to give conditions on the structure of observers of positive systems that always guarantee nonnegative estimates of the states, and tract asymptotically the actual state. Since, we have seen that the error resulting form the estimate of the actual state is also a solution to the linear time-delay system (3), we need a stability result to guarantee the convergence of the error to zero. In the following, conditions for the global asymptotic stability of a general positive linear time-delay system (5) are presented.

Theorem 2.1 *Assume that System (5) is positive, or equivalently that the matrix A is Metzler and $A_1, \ldots, A_m$ are positive matrices. Then, the following statements are equivalent.*

- *(i): There exist an initial condition $\phi^*(\cdot)$ taking values in the interior of $\Re^n_+$ such that System (5) is asymptotically stable.*
- *(ii): System (5) is asymptotically stable for every initial condition $\phi(\cdot)$ taking values in $\Re^n_+$.*
- *(iii): System (5) is asymptotically stable for every initial condition $\phi(\cdot)$ taking values in $\Re^n$ ($\phi(\cdot)$ has indefinite sign).*
- *(iv): There exists $\lambda \in \Re^n$ such that*

$$(A + \sum_{i=1}^{m} A_i)\lambda < 0, \ \ \lambda > 0. \qquad (6)$$

- *(v): $A + \sum_{i=1}^{m} A_i$ is a Hurwitz matrix: the real part of its eigenvalues is strictly negative.*

Proof: The implications $(iii) \Rightarrow (ii) \Rightarrow (i)$ are obvious.
$(iii) \Rightarrow (iv)$: By integrating System (5) we have

$$x(T) - x(0) = A \int_0^T x dt + \sum_{i=1}^{m} A_i \int_0^T x(t - \tau_i) dt,$$

which by change of variable can be expressed as the following identity

$$(A + \sum_{i=1}^{m} A_i) \int_0^T x dt =$$
$$x(T) + \sum_{i=1}^{m} A_i \int_T^{T-\tau_i} x(t) dt - \sum_{i=1}^{m} A_i \int_{-\tau_i}^{0} \phi^*(t) dt - x(0),$$

since $x(T)$ goes to zeros, then also $\sum_{i=1}^{m} A_i \int_T^{T-\tau_i} x(t)dt$.
Moreover, since ϕ^* is positive the term $\sum_{i=1}^{m} A_i \int_{-\tau_i}^{0} \phi^*(t)dt+$ $x(0)$ is constant and positive. Then by gathering these facts with a sufficiently large T we obtain

$$(A + \sum_{i=1}^{m} A_i)\lambda < 0, \ \ \lambda > 0,$$

where λ is defined as $\lambda = \int_0^T x dt$.
$(i) \Rightarrow (ii)$: Assume that for a particular initial functional condition $\phi^*(\cdot) > 0$, System (5) is asymptotically stable. Let $\phi(\cdot) \geq 0$ be any initial functional condition. Then, there exist a positive scalar $\alpha > 0$ such that

$$\phi(s) \leq \alpha \phi^*(s), \ \ \forall s : -\tau \leq s \leq 0.$$

Now, by using Lemma 2.2 we can conclude

$$0 \leq x(t) \leq \alpha x^*(t), \ \ \forall t \geq 0,$$

where $x(\cdot)$ and $x^*(\cdot)$ are respectively the associated trajectories to $\phi(\cdot)$ and $\phi^*(\cdot)$. Since $x^*(t)$ goes to zeros then also $x(t)$. Hence, we have shown that System (5) is asymptotically stable for every initial functional condition $\phi(\cdot)$ taking values in $\Re^n_+$.
$(ii) \Rightarrow (iii)$: This implication results from the linearity of the system and the fact that ϕ can be decomposed as $\phi = \phi^+ - \phi^-$ where $\phi^+ \geq 0, \phi^- \geq 0$.
$(iv) \Rightarrow (v)$: it suffices to apply the well-known Perron-Frobenius theorem.
The rest of the proof is similar to the one in [9, 4, 2].
$\diamondsuit$

Remark 2.1 *if $A + \sum_{i=1}^{m} A_i$ is a Metzler and Hurwitz matrix, then its transpose is also is a Metzler and a Hurwitz matrix. That is condition (iv) in Theorem 2.1 is also equivalent to its dual [9]:*
There exists $\beta \in \Re^n$ such that

$$(A^T + \sum_{i=1}^{m} A_i^T)\beta < 0, \ \ \beta > 0. \qquad (7)$$

3 Positive observers design

In this section, the objective is to give conditions on the structure of observers of positive time-delay systems that always guarantee nonnegative estimates of the states, and which tract asymptotically the actual state. In particular, we will show that this problem can be cast as an LP problem and even with indefinite sign outputs of the actual system (not necessary positive). In addition to the asymptotic convergence of the error e, we need to guarantee the nonnegativity of observer (3). This nonnegativity condition follows from additional restrictions on the structure of its dynamic. Such restrictions are provided by the following.

Lemma 3.1 *There exists a positive observer (3) for positive system (1) if and only if the following holds true.*

(i): $A - LC$ is a Metzler matrix and $A_1 - LC_1 \geq 0, \ldots, A_m - LC_m \geq 0$.

(ii): $LC \geq 0$ and $LC_1 \geq 0, \ldots, LC_m \geq 0$.

(iii): $A - LC + \sum_{i=1}^{m}(A_i - LC_i)$ is a Hurwitz matrix.

Proof: First, assume that there exists a positive observer of form (3). Since the error associated to the observed state $e(\cdot) \doteq \hat{x}(\cdot) - x(\cdot)$ must converge to zero and since this error is also a solution to the linear delayed system

$$\frac{d\hat{e}}{dt} = (A - LC)\hat{e} + \sum_{i=1}^{m}(A_i - LC_i)\hat{e}(t - \tau_i),$$

then by Theorem 2.1 we conclude that $A - LC + \sum_{i=1}^{m}(A_i - LC_i)$ must be a Hurwitz matrix. Moreover, regarding to the fact that the following augmented system is positive

$$\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & 0 \\ LC & A - LC \end{bmatrix}\begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \sum_{i=1}^{m}\begin{bmatrix} A_i & 0 \\ LC_i & A_i - LC_i \end{bmatrix}\begin{bmatrix} x(t-\tau_i) \\ \hat{x}(t-\tau_i) \end{bmatrix},$$

then by using Lemma 2.1 we conclude that necessarily conditions (i) and (ii) are fulfilled. The sufficiency part of the proof follows the same line of argument. $\diamondsuit$ In connection with the result of Lemma 3.1, the following relevant remark is in order.

Remark 3.1 *In already published paper without the presence of delays [17], the gain of the observer is shown to be positive $L \geq 0$. However, in the multi-input case, the positivity of the gain is only sufficient condition. But, for the single-input system with positive matrix C this condition is necessary and sufficient. More precisely, in the multi-input case this condition is only sufficient when the matrix C is positive. In the following counterexample we show that L is positive is not necessary (even with C positive) for the existence of a positive observer.*

Counterexample: Consider a linear system without delays as follows: $A = \begin{bmatrix} -6 & 3 & 2 \\ 2 & -8 & 3 \\ 3 & 4 & -7 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $L = \begin{bmatrix} 2 & -1 \\ 2 & -1 \\ 2 & -1 \end{bmatrix}$.
Even the matrix L is a non positive gain it does exist a positive observer associated to L. First, by calculation the matrix L fulfills the condition: $LC \geq 0$ and $A - LC$ is Metzler. Thus, according to Lemma 3.1 the observer with the given gain L is positive. Also, $A - LC$ is a Hurwitz matrix. This follows from the result of Theorem 2.1, since by choosing $\lambda^T = \begin{bmatrix} 1 & 1 & 3 \end{bmatrix} > 0$, then we have $(A - LC)\lambda = \begin{bmatrix} -2 \\ -2 \\ -19 \end{bmatrix} < 0$ (Alternatively, one may check that the spectrum of $(A - LC)$ is $-4.4020, -8.6628, -10.9352$).

Positive observers possess an inherent nice property. In fact, they can be used to derive upper and lower bounds on the observed states if the initial functional condition of the actual system is unknown but bounded.

Theorem 3.1 *Assume that the initial condition $\phi(\cdot)$ of the observed system is unknown but bounded:*

$$0 \leq \underline{x} \leq \phi(t) \leq \overline{x}, \ \textit{for} \ -\tau \leq t \leq 0.$$

Then, the evolution of the current state function $x(\cdot)$ will always be between the estimated states $\hat{x}_{lower}(\cdot)$ and $\hat{x}_{upper}(\cdot)$:

$$\hat{x}_{lower}(t) \leq x(t) \leq \hat{x}_{upper}(t), \ \forall t \geq 0,$$

where $\hat{x}_{lower}(\cdot)$ has as initial condition $\phi_{lower}(\cdot) \equiv \underline{x}$ and $\hat{x}_{upper}(\cdot)$ has as initial condition $\phi_{upper}(\cdot) \equiv \overline{x}$. Moreover, these estimated bounds are positive and converge asymptotically to the actual state.

Proof: Effectively, since the error is a solution of a positive time-delay system (4), then it suffices to utilize the monotonicity of the error and this can be done via Lemma 2.2 $\diamondsuit$

Remark 3.2 *System (1) is said to be compartmental if the associated matrix A is Metzler and*

$$\sum_{i=1}^{n}[A + \sum_{k=1}^{m} A_k]_{ij} \leq 0.$$

We have shown that the positive observer must satisfy the condition $LC \geq 0$ and $LC_1 \geq 0, \ldots, LC_m \geq 0$. It is easily seen that if system (1) is compartmental then the associated observer is also compartmental. This fact was also pointed out in [17] for positive linear systems without delays.

The following result provides necessary and sufficient conditions for the existence of positive time-delay observers. Also, it provides a computational approach via linear programming (our conditions are expressed as an LP).

Theorem 3.2 *The following statements are equivalent*

(i): There exists a positive observer of system (1) of the form:

$$\frac{d\hat{x}}{dt} = (A - LC)\hat{x} + \sum_{i=1}^{m}(A_i - LC_i)\hat{x}(t-\tau_i) + Ly,$$
$$\hat{x}(t) \geq 0, \quad \forall t \geq 0.$$

(ii): There exists a matrix $L \in \Re^{n\times r}$ *such that* $LC_1 \geq 0, \dots, LC_m \geq 0$ *and* $A - LC + \sum_{i=1}^{m}(A_i - LC_i)$ *is a Metzler and Hurwitz matrix.*

(iii): The following LP problem in the variables $\lambda \in \Re^n$ *and* $Z \in \Re^{r\times n}$ *is feasible*

$$\left\{\begin{array}{l} (A^T + \sum_{i=1}^{m} A_i^T)\lambda - (C + \sum_{i=1}^{m} C_i^T)Z \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} < 0, \\ \lambda > 0, \\ A^T \mathbf{diag}(\lambda) - C^T Z + I \geq 0, \\ C^T Z \geq 0, \\ C_i^T Z \geq 0, \;\; 1 \leq i \leq n, \\ A_i^T \mathbf{diag}(\lambda) - C_i^T Z \geq 0, \;\; 1 \leq i \leq n. \end{array}\right. \tag{8}$$

Moreover, a gain matrix L in the statements (i) and (ii) can be computed as follows

$$L = \mathbf{diag}(\lambda)^{-1} Z^T,$$

where the vector λ *and the matrix* Z *are any feasible solution to the above LP problem.*

Proof: The equivalence between (i) and (ii) is straightforward from Lemma 3.1.
Now let us show that (ii) and (iii) are equivalent. First, note that $A - LC + \sum_{i=1}^{m}(A_i - LC_i)$ is Metzler and Hurwitz if and only if its transpose is Metzler and Hurwitz. Thus it follows that (ii) is equivalent to the existence of a matrix $L \in \Re^{n\times r}$ satisfying the following conditions:

$$\left\{\begin{array}{l} A^T - C^T L^T \text{ is Metzler }, \\ C^T L^T \geq 0, C_1^T L^T \geq 0, \dots, C_m^T L^T \geq 0, \\ A^T - C^T L^T + \sum_{i=1}^{m}(A_i^T - C_i^T L^T) \text{ is Hurwitz }, \\ A_1^T - C_1^T L^T \geq 0, \dots, A_m^T - C_m^T L^T \geq 0. \end{array}\right.$$

To show that condition (iii) is equivalent to the above conditions, it suffices to define $L = \mathbf{diag}(\lambda)^{-1} Z^T$. Then, note that $A^T - C^T L^T$ is Metzler if and only if $(A^T - C^T L^T)\mathbf{diag}(\lambda)$ is Metzler, or

$$(A^T - C^T L^T)\mathbf{diag}(\lambda) + I \geq 0,$$

by choosing λ with sufficiently small components (since the stability condition is homogeneous in λ). The above inequality is nothing else than the condition

$$A^T \mathbf{diag}(\lambda) - C^T Z + I \geq 0,$$

in the LP constraints (8). The rest of the proof is a simple matrix manipulation as shown above. ◊

4 Stabilization does not hold

In the preceding section, we have shown how to solve the observation problem in terms of LP. In contrast with the classical theory of observers, it is shown here that the stabilization of positive systems by using positive extended Luenberger observers is impossible. More precisely, consider the following forced linear system:

$$\begin{aligned} \frac{dx}{dt} &= Ax + \sum_{i=1}^{m} A_i x(t-\tau_i) + Bu, \\ y(t) &= Cx(t) + \sum_{i=1}^{m} C_i x(t-\tau_i) \in \Re^r, \end{aligned} \tag{9}$$

where the trajectory $x \in \Re^n_+$ is assumed to be positive under the input signal $u \in \Re^p$, but its evolution is unknown. The output $y \in \Re^r$ is known and not necessarily positive (C may be indefinite sign matrix). Now, our objective here is to show that there does not exists any positive observer ($\hat{x} \geq 0$) in the extended Luenberger form:

$$\frac{d\hat{x}}{dt} = (A - LC)\hat{x} + \sum_{i=1}^{m}(A_i - LC_i)\hat{x}(t-\tau_i) + Ly + Bu, \tag{10}$$

which, simultaneously, guaranties the positivity of the observed state $x \geq 0$, and stabilizes asymptotically system (9), when an observer feedback law $u = K\hat{x}$ is utilized. To show this, we provide the following result.

Theorem 4.1 *Assume that the unforced system (9) (with* $u = 0$*) is not asymptotically stable, or equivalently* $A + \sum_{i=1}^{m} A_i$ *is not a Hurwitz matrix. Then, the separation principle fails. More precisely, there does not exist a positive observer of the form (10) for system (9) together with a feedback law* $u = K\hat{x}$ *that is both asymptotically stabilizing and guaranties the non-negativity of the observed state.*

Proof: Assume that $A + \sum_{i=1}^{m} A_i$ is not Hurwitz and that there exist matrices $L \in \Re^{n\times r}$, $K \in \Re^{p\times n}$ which fulfill the positivity of x and $\hat{x}$, and their asymptotic convergence to zeros, under the feedback control $u = K\hat{x}$. We now show that such statement leads to a contradiction:
Since the following augmented system must be positive:

$$\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & BK \\ LC & A - LC + BK \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} A_i & 0 \\ LC_i & A_i - LC_i \end{bmatrix} \begin{bmatrix} x(t-\tau_i) \\ \hat{x}(t-\tau_i) \end{bmatrix}, \tag{11}$$

then by using Lemma 2.1 it is easy to see that necessarily $BK \geq 0$ holds.
Now, by using the following transformation: $v_1 = x - \hat{x}$ and $v_2 = \hat{x}$, then one can check that the augmented system (11) is similar to the following time-delay system:

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} A - LC & 0 \\ LC & A + BK \end{bmatrix}\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} + \sum_{i=1}^{m}\begin{bmatrix} A_i - LC_i & 0 \\ LC_i & A_i \end{bmatrix}\begin{bmatrix} v_1(t-\tau_i) \\ v_2(t-\tau_i) \end{bmatrix}. \tag{12}$$

Henceforth, if the augmented system (11) is asymptotically stable, then necessarily $A + \sum_{i=1}^{m} A_i + BK$ must be a Hurwitz matrix, but this is impossible. As we have shown before that $BK \geq 0$, this necessarily implies that $A + \sum_{i=1}^{m} A_i$ is Hurwitz. Hence, the fact that $A + \sum_{i=1}^{m} A_i + BK$ is a Hurwitz positive matrix leads to the contradiction that the unforced system (9) (with $u = 0$) is asymptotically stable. To see this, Theorem 2.1 implies the existence of $\lambda > 0$ such that $(A + \sum_{i=1}^{m} A_i + BK)\lambda < 0$. Thus, as BK is positive, necessarily, $(A + \sum_{i=1}^{m} A_i)\lambda < 0$. Then, by using Theorem 2.1, we have shown that $A + \sum_{i=1}^{m} A_i$ is necessarily a Hurwitz matrix, or equivalently that the unforced system (9) is asymptotically stable. This fact completes the proof. $\diamondsuit$

5 Conclusions

We have proposed to study a generalization of the classical theory of observers by investigating for time-delay positive systems, the existence and characterization of positive observers, that is observers that define themselves a positive linear system. In addition to developing theoretical results on the solvability of such problems, we have complemented such analysis by simply using a very efficient technique: Linear Programming (LP). All the proposed conditions are necessary and sufficient, which turn out to be solvable in terms of LP.

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