On the stability of positive nonlinear systems:

Cooperative and concave system dynamics with applications to distributed networks

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Abstract

In this thesis we discuss the existence and stability properties for positive nonlinear systems on $\mathbb{R}^n$. In general, a positive system is a system whose evolution remains in the positive orthant for all future time, given a positive initial condition. We consider in particular systems suitable to describe cooperative dynamics, that is, in which the trajectory's evolution is monotone in the forward time direction, often encountered in biology, economics and engineering sciences.

The most general class of systems we consider is associated to sub-homogeneous vector fields, which includes as a special case concave vector fields. Conditions on the existence and uniqueness of an equilibrium point in the interior of the positive orthant are given. Under the conditions for existence of a fixed point, the stability properties of the system are characterized and an estimate of the domain of attraction is made. A comparison with the class of so-called standard interference functions, available in the wireless network power control literature, is made. In particular, we examine similarities and differences with interference functions in which convergence is achieved through a contractivity property. The scalability property of standard interference functions is equivalent to strict sub-homogeneity, therefore, as a special case of the main theorems, conditions for the convergence of many distributed power control laws are given.

In general we consider the case in which the Jacobian matrix of the systems' vector field is irreducible, that is the associated influence digraph is strongly connected. When irreducibility does not hold, our main theorems are still valid given that the system is distributed and each isolated subsystem with no incoming edges satisfies the same conditions given by the theorems for irreducible systems. Finally, we give some simple examples and simulations proving the concreteness of our results.

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Introduction

1.1 Introduction

A system is said to be positive if the evolution of the trajectory from any nonnegative initial condition is nonnegative for all future time, in other words Rn + is positively invariant. Examples of intrinsic positive systems can be found for example in biology, ecology and communication. In biology, the states of the biological model typically represent concentrations of chemical species, while in ecology they represent populations. In communication, and specifically in wireless networks theory, the states represent the power needed for transmission ([12, 14, 30]).

Our attention will be on cooperative systems, that is, systems for which the trajectory's evolution is monotone in the forward time direction.

The main result in this thesis is to give conditions for the existence and uniqueness of a fixed-point for the class of monotone concave vector fields and, more generally, monotone sub-homogeneous vector fields. Conditions can be given that guarantee convergence to the origin (similar to the diagonal dominance of [5]), as well as convergence to a strictly positive equilibrium point. The latter conditions are much more general than those available in the literature for nonlinear positive systems [9]. For the vector fields considered in this work such conditions are global (in $\mathbb{R}^n_+$). These results are derived using the Perron-Frobenius theory for nonnegative matrices. In particular, we show that the conditions we obtain can be rephrased in terms of the spectral radius of the Jacobian of the system at different points of the positive orthant. Both weakly and strongly connected influence digraphs cases associated to the Jacobian matrix of the vector fields are also considered, which correspond to reducible and irreducible associated networks, respectively.

In this thesis, the focus is on a particular class of nonlinear positive systems. This class is mainly motivated by distributed power control laws for wireless networks, such as in [14],

but it is applicable to more general problems. A well accepted framework in literature for studying power control problems was proposed in [30]. It is called standard interference functions framework, and it includes linear and several important nonlinear power control laws. Interference here refers to the effect of a number of end-user devices (e.g. mobile phones) that are trying to transmit together and therefore force each of them to use a certain amount of power in the transmission, in order to overcome their mutual interference. For each device, the aim is to create a closed-loop system which is positive (the state being a power it cannot become negative), and has a positive equilibrium point.
One of the main results in this manuscript shows how the scalability property of standard interference functions, which is equivalent to strict sub-homogeneity, together with some other conditions demonstrated later, guarantees the existence of a unique fixed point and its asymptotic stability. Various extensions of the basic framework have been proposed in the literature, the most prominent being those by [3, 28] and [12]. In particular, the contractive interference functions introduced in [12] guarantee existence and uniqueness of a fixed point along with convergence. In the thesis, the class of contractive interference functions is characterized in terms of spectral radius. It will be shown that this class implies that the spectral radius in each point of the state space must be less than 1. The properties of contractive interference functions can therefore be obtained a special case from the results demonstrated in this thesis.

1.2 Preliminaries

Throughout this paper let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}^n$ be the space of column vectors of size n with real elements and $\mathbb{R}^{n\times n}$ be the space of $n\times n$ matrices with real entries. For $x\in\mathbb{R}^n$ and $i=1,\dots,n$, x_i denotes the i^{th} coordinate of x. Similarly if $A\in\mathbb{R}^{n\times n}$ then a_{ij} or $[A]_{ij}$ denotes the element in position (i,j); $\Lambda(A)$ denotes the spectrum of A and $\rho(A)$ denotes its spectral radius, i.e.

$$\rho(A) = max\{|\lambda|, \lambda\in\Lambda(A)\} \tag{1.1}$$

its spectral abscissa as

$$\mu(A) = max\{Re(\lambda), \lambda\in\Lambda(A)\}. \tag{1.2}$$

The positive orthant of $\mathbb{R}^n$ is defined as $\mathbb{R}^n_+ \triangleq \{x\in\mathbb{R}^n : x_i\geq 0,\ 1\leq i\leq n\}$. The interior of $\mathbb{R}^n_+$ is defined as $int\left(\mathbb{R}^n_+\right) \triangleq \{x\in\mathbb{R}^n : x_i > 0,\ 1\leq i\leq n\}$. The boundary of $\mathbb{R}^n_+$ is defined as $bd\left(\mathbb{R}^n_+\right) \triangleq \mathbb{R}^n_+\backslash int\left(\mathbb{R}^n\right)$.
For vectors $x,y\in\mathbb{R}^n$ we write $x\geq y$ if $x_i\geq y_i$ for $i=\{1,\dots,n\}$; $x>y$ if $x_i>y_i$ for $i=$

$\{1,\ldots,n\}$; Analogously we say a matrix $A \in \mathbb{R}^{n\times n}$ to be non-negative (positive) if $a_{ij} \geq 0$ ($a_{ij} > 0$) for all couples (i,j). Furthermore $\mathbb{1}^T \triangleq (1,\ldots,1)$.

Definition 1.2.1. *A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be non-decreasing if*

$$x \leq y \Rightarrow f(x) \leq f(y) \tag{1.3}$$

for all $x, y \in \mathbb{R}^n$. It is said to be increasing if

$$x \leq y \Rightarrow f(x) \leq f(y) \text{ and } x < y \Rightarrow f(x) < f(y) \tag{1.4}$$

for all $x, y \in \mathbb{R}^n$.

Proposition 1.2.1. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a non-decreasing (or increasing) function then the Jacobian matrix of f is non-negative, i.e.,*

$$\frac{\partial f}{\partial x}(x) \geq 0, \quad \forall x \in \mathbb{R}^n \tag{1.5}$$

Proof. Proof is straightforward and will be omitted. ∎

1.2.1 Concave vector fields

In this paper we focus mainly on concave vector fields, thus we need the following definitions and results found in Boyd and Vandenberghe [6].

Definition 1.2.2. *Let $\mathscr{D}$ be a convex subset of $\mathbb{R}^n$. A vector field $f : \mathscr{D} \to \mathbb{R}^n$ is said to be convex if*

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \tag{1.6}$$

for all $x, y \in \mathscr{D}$ and for $\alpha \in [0,1]$. It is said to be strictly convex if the inequality in (1.6) holds strictly for $\alpha \in (0,1)$ and $x \neq y$.
A vector field f is said to be concave if -f is convex, i.e.

$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha) f(y) \tag{1.7}$$

for all $x, y \in \mathscr{D}$ and for $\alpha \in [0,1]$. It is said to be strictly concave if the inequality in (1.7) holds strictly for $\alpha \in (0,1)$ and $x \neq y$.

A characterization of concave vector fields can be given looking at their Jacobian matrix. The condition means that the tangent plane in any point must always stay on top of the function.

Proposition 1.2.2. *A differentiable vector field $f : \mathscr{D} \to \mathbb{R}^n$ is concave if and only if*

$$f(y) \leq f(x) + \frac{\partial f}{\partial x}(x)(y-x) \tag{1.8}$$

for all $x, y \in \mathscr{D}$. Strictly concave if the inequality holds strictly for all $x \neq y$.

Remark. *The affine function of y given by $f(x) + \frac{\partial f}{\partial x}(x)(y-x)$ is the first-order Taylor approximation of f near x. The interpretation of the inequality* (1.8) *is as follows: for a concave vector field, the first-order Taylor approximation is an over-estimator of the vector field f in $\mathscr{D}$. Conversely, if the first-order Taylor approximation of a function is always an over-estimator of the vector field f in $\mathscr{D}$, then the function is concave.*

Definition 1.2.3. *Let $\mathscr{D}$ be a convex subset of $\mathbb{R}^n$. A vector field $f : \mathscr{D} \to \mathbb{R}^n$ is said to be subhomogeneous of degree $\tau > 0$ if*

$$f(\alpha x) \geq \alpha^\tau f(x) \tag{1.9}$$

for all $x \in \mathscr{D}$ and for $\alpha \in [0,1]$. It is called strictly subhomogeneous if inequality holds strictly for $\alpha \in (0,1)$.

Let us show the relationship between concave vector fields and subhomogeneous vector fields.

Proposition 1.2.3. *Let $\mathscr{D}$ be a convex subset of $\mathbb{R}^n$, with $0 \in \mathscr{D}$. Let $f : \mathscr{D} \to \mathbb{R}^n$ be a vector field such that $f(0) \geq 0$. If f is concave then f is subhomogeneous of degree* 1.

Proof. Since f is concave the following holds

$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha) f(y) \tag{1.10}$$

for all $x, y \in \mathscr{D}$ and for $\alpha \in [0,1]$. By choosing $y = 0$ we have

$$f(\alpha x) \geq \alpha f(x) + (1-\alpha) f(0) \tag{1.11}$$

since $(1-\alpha) f(0) \geq 0$ we obtain $f(\alpha x) \geq \alpha f(x)$ which is exactly the definition in (1.9) with $\tau = 1$. ■

It can be easily seen from Proposition 1.2.3 that strict concavity of f and $f(0) \geq 0$ implies strict subhomogeneity of degree 1. The following lemma can be found in [5]:

Lemma 1.2.1. *The vector field $f : \mathscr{D} \to \mathbb{R}^n$ is subhomogeneous of degree $\tau > 0$ if and only if*

$$\frac{\partial f}{\partial x}(x)x \leq \tau f(x), \quad \forall x \geq 0 \tag{1.12}$$

Strictly subhomogeneous if the inequality holds strictly for all $x > 0$.

Proof. We first note that f is subhomogeneous of degree τ if and only if for any $x \geq 0$, the mapping

$$\lambda \to \lambda^{-\tau} f(\lambda x)$$

is non-increasing for all $\lambda > 0$.
Let $x \geq 0$ be given. If f is subhomogeneous then for any $\mu \geq \lambda$ we have

$$f(\lambda x) = f\left(\frac{\lambda}{\mu}\mu x\right) \geq \left(\frac{\lambda}{\mu}\right)^{\tau} f(\mu x) \tag{1.13}$$

which implies

$$\lambda^{-\tau} f(\lambda x) \geq \mu^{-\tau} f(\mu x) \tag{1.14}$$

Thus, $\lambda^{-\tau} f(\lambda x)$ is a non-decreasing function with respect to λ for all $\lambda > 0$.
Differentiating with respect to λ, we see that f is subhomogeneous if and only if for all $\lambda > 0$

$$\frac{d}{d\lambda}(\lambda^{-\tau} f(\lambda x)) \leq 0 \Leftrightarrow -\tau\lambda^{-\tau-1} f(\lambda x) + \lambda^{-\tau}\frac{\partial f}{\partial x}(\lambda x)x \leq 0$$

Rearranging this last inequality, we see that f is subhomogeneous if and only if

$$\frac{\partial f}{\partial x}(\lambda x)\lambda x \leq \tau f(\lambda x) \quad \forall x \geq 0, \ \forall \lambda > 0$$

This last statement is equivalent to

$$\frac{\partial f}{\partial x}(x)x \leq \tau f(x) \quad \forall x \geq 0 \tag{1.15}$$

This concludes the proof for the subhomogeneous case. With slight changes, this same proof holds for the case of strict subhomogeneity. ■

1.2.2 Non-negative matrices

We will now give some standard definitions and results for non-negative matrices from Berman and Plemmons [2], Fornasini [13].

Definition 1.2.4. *Let $A \in \mathbb{R}^{n \times n}$ be a non-negative matrix. A is irreducible if for every $i, j \in \{1, \dots, n\}$ there exists an exponent k such that*

$$[A^k]_{ij} > 0.$$

Definition 1.2.5. *Let $A \in \mathbb{R}^{n \times n}$ be a non-negative matrix. We can define its influence graph as a directed graph with n vertices $x_1, \dots x_n$ such that there exists a weighted edge from x_j to x_h whenever $a_{hj} \neq 0$ with weight a_{hj}.*

The influence graph is *strongly connected* if there exists a directed path between node x_r and node x_s for all $r, s \in \{1, \dots, n\}, r \neq s$. It is *weakly connected* if there exists an undirected path between node x_r and node x_s for all $r, s \in \{1, \dots, n\}, r \neq s$
We can now give the following characterization of irreducibility.

Proposition 1.2.4. *Let $A \in \mathbb{R}^{n \times n}$ be a non-negative matrix. A is irreducible if and only if any of the following conditions hold*

1. *(definition) for all couple (i, j) their exists an exponent k such that $[A^k]_{ij} > 0$*

2. *the influence graph of A is strongly connected*

3. *there do not exist any permutation matrix Π such that*

$$\Pi^T A \Pi = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

where A_{11} and A_{22} are two non-empty square matrices.

The following, known as Perron-Frobenius theorem, also holds. Some of the proofs in the next chapters heavily relies on this.

Theorem 1.2.1 (Perron-Frobenius Theorem)**.** *If $A \in \mathbb{R}^{n \times n}$ is an irreducible non-negative matrix then*

1. *there exists a real number $\lambda_0 > 0$ and a vector $v_0 > 0$ such that $Av_0 = \lambda_0 v_0$*

2. *for every other $\lambda \in \Lambda(A)$ it follows $|\lambda| \leq \lambda_0$, i.e $\rho(A) = \lambda_0$.*

3. *if* $\bar{A} \geq A$ *and* $\bar{A} \neq A$*, then* $\bar{\lambda}_0 > \lambda_0$

Remark. *Condition 3 states that it is sufficient to have at least one different element to have a larger Perron-Frobenius eigenvalue, i.e.* $\bar{a}_{ij} \geq a_{ij}$ *and* $\exists\ \bar{a}_{ij} > a_{ij}$ *implies* $\bar{\lambda}_0 > \lambda_0$.

Definition 1.2.6. *A real* $n \times n$ *matrix A is Metzler if and only if* $a_{ij} \geq 0$, $\forall i \neq j$*, i.e. its off-diagonal entries are non-negative.*

For irreducible Metzler matrices first point of Theorem 1.2.1 also holds. The proof can be found in Horn [15].
A Metzler matrix A is irreducible if and only if for every non-empty proper subset K of $N := \{1,\dots,n\}$, there exists an $i \in K$, $j \in N \backslash K$ such that $a_{ij} \neq 0$. A reducible non-negative matrix can be transformed into its normal form.

Proposition 1.2.5 (Reducible normal form)**.** *Let* $A \in \mathbb{R}^{n\times n}$ *be a reducible non-negative matrix. Then there exists a permutation matrix* Π *that transforms A into its normal form, i.e.*

$$\bar{A} = \Pi^T A \Pi = \left[\begin{array}{cccc|cccc} \bar{A}_{1,1} & & & & & & & \\ 0 & \bar{A}_{2,2} & & & & 0 & & \\ \vdots & \dots & \ddots & & & & & \\ 0 & 0 & \dots & \bar{A}_{h,h} & & & & \\ \hline \star & \star & \dots & \star & \bar{A}_{h+1,h+1} & & & \\ \star & \star & \dots & \star & \star & \bar{A}_{h+2,h+2} & & \\ \star & \star & \dots & \vdots & \vdots & \dots & \ddots & \\ \star & \star & \dots & \star & \star & \star & \dots & \bar{A}_{kk} \end{array}\right] \tag{1.16}$$

where every square block $\bar{A}_{i,i}$, $i = 1,\dots,k$ *is either an irreducible matrix or a* 1×1 *zero matrix. Furthermore if* $h < k$ *on every line from the* $(h+1)^{th}$ *forward at least one of the off-diagonal matrices* $\star$ *has a positive element. The blocks* $\bar{A}_{i,i}$, $i = 1,\dots,h$ *are called isolated blocks.*

Proof. For proof see Fornasini [13]. ∎

1.3 Stability and Cooperativity

In this paper we are interested in studying the stability properties of an equilibrium point. We now give some other basic definitions. Given a system

$$\dot{x} = f(x), \quad x(0) = x_0 \tag{1.17}$$

The forward solution with initial condition $x_0 \in \mathbb{R}^n$ at $t = 0$ is denoted as $x(t,x_0)$ and is defined on the maximal forward interval of existence $[0,t_{max}(x_0))$, for us $t_{max}(x_0) = \infty$. A set $\Omega \subset \mathbb{R}^n$ is called forward invariant if and only if for all $x_0 \in \Omega$, $x(t,x_0) \in \Omega$ for all $t \in [0,t_{max}(x_0))$.
A point x^* is said to be an *equilibrium point* of (1.17) if it has the property that whenever the state of the system starts at x^* it will remain at x^* for all future time. That is $x(t,x^*) = x^*$, for all $t \geq 0$.
As already stated we are interested in positive systems and we are interested in finding the conditions that guarantees its positiveness.. The system (1.17) is to be *positive* if $x(t,x_0) \in \mathbb{R}^n_+$ for all $x_0 \in \mathbb{R}^n_+$ and for all $t \geq 0$, i.e. $\mathbb{R}^n_+$ is forward invariant. As shown in De Leenherr [10] and assuming the uniqueness of solutions of the system, the following property is a necessary and sufficient condition for positivity of the system:

$$x_i = 0 \Rightarrow f_i(x) \geq 0, \quad \forall x \in bd(\mathbb{R}^n_+) \tag{1.18}$$

In the remainder of this paper we focus on cooperative systems.

Definition 1.3.1 (Cooperativity)**.** *System* (1.17) *is said to be cooperative in $\mathscr{D} \subset \mathbb{R}^n$ if the differentiable vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is such that the Jacobian matrix $\frac{\partial f}{\partial x}(x)$ is Metzler for all $x \in \mathscr{D}$.*

Definition 1.3.2 (Monotonicity)**.** *The system in* (1.17) *is said to be monotone if for all $x_0, y_0 \in \mathbb{R}^n_+$ we have*

$$x_0 \leq y_0 \Rightarrow x(t,x_0) \leq x(t,y_0) \textit{ for all } t \geq 0$$

The following condition is an easy way to verify the monotonicity of a system (see Kamke [16]).

Definition 1.3.3 (Kamke Condition)**.** *The vector field $f : \mathscr{D} \to \mathbb{R}^n$ defined on an open subset $\mathscr{D}$ of $\mathbb{R}^n$ is said to be of* type K *or satisfy* Kamke Condition*, if for each i, $f_i(a) \leq f_i(b)$ for any two points a and b in $\mathscr{D}$ satisfying $a \leq b$ and $a_i = b_i$.*

The following Proposition shows the relationship between a vector field satisfying the Kamke Condition and monotonicity.

Proposition 1.3.1. *Let f be type K in an open subset $\mathscr{D}$ of $\mathbb{R}^n$. The system* (1.17) *is monotone.*

Proof. Proof can be found in Smith [24], Proposition 3.1.1. ■

In Smith [24] it is shown that cooperative systems are monotone. A subset $\mathscr{D}$ of $\mathbb{R}^n$ is said to be convex if $\alpha x + (1-\alpha)y \in \mathscr{D}$ for all $\alpha \in [0,1]$ and $x,y \in \mathscr{D}$. Let $f : \mathscr{D} \to \mathbb{R}^n$ be

cooperative.
Then the fundamental theorem of calculus implies that f is of *type K* in $\mathscr{D}$. In fact, if $a \leq$ and $a_i = b_i$ then

$$f_i(b) - f_i(a) = \int_0^1 \sum_{i \neq j} \frac{\partial f_i}{\partial x_j}(a + r(b-a))(b_j - a_j)dr \geq 0 \tag{1.19}$$

which is greater than or equal to zero because from the definition of cooperativity of f, the Jacobian matrix is Metzler for all $r \in [0,1]$.

Using some of the previous results, we can relate the monotonicity of the system to the non-decreasing property of its vector field.

Proposition 1.3.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable vector field. If f is non-decreasing then system* (1.17) *is monotone.*

Proof. Since f is non-decreasing Proposition 1.2.1 holds, yielding

$$\frac{\partial f}{\partial x}(x) \geq 0, \quad \forall x \in \mathbb{R}^n \tag{1.20}$$

therefore Metzler. This implies that system (1.17) is cooperative, thus monotone. This concludes the proof. ∎

An important property of monotone systems that we will use in the next chapters to prove the convergence of the system to an equilibrium point is the following lemma (see Bokharaie [4]):

Lemma 1.3.1. *Let $\mathscr{D}$ be an open subset of $\mathbb{R}^n$ and let $f : \mathscr{D} \to \mathbb{R}^n$ be a cooperative vector field. Assume there exists a vector w such that $f(w) < 0$ ($f(w) > 0$). Then the trajectory $x(t,w)$ of system* (1.17) *is decreasing (increasing) for $t \geq 0$. In the case of $f(w) \leq 0$ ($f(w) \geq 0$) the trajectory will be non-increasing (non-decreasing).*

Proof. Proof can be found in Smith [24], Proposition 3.2.1. ∎

We now give some other fundamental definitions:

Definition 1.3.4. *An equilibrium point $x^* \in \mathbb{R}^n$ of system* (1.17) *is*

- *stable if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\|x_0 - x^*\| < \delta \Rightarrow \|x(t,x_0) - x^*\| < \varepsilon, \quad \forall t \geq 0$$

- *unstable if not stable*
- *asymptotically stable if it is stable and* $\exists\,\delta > 0$ *such that the following holds for* $x_0 \in \mathbb{R}^n$

$$\|x_0 - x^*\| < \delta \Rightarrow \lim_{t\to\infty} x(t, x_0) = x^*$$

Definition 1.3.5. *Let* x^* *be an equilibrium point. The set*

$$A(x^*) = \{x_0 \in \mathbb{R}^n : x(t, x_0) \to x^* \text{ as } t \to \infty\}$$

is called domain of attraction of x^*.

Chapter 2

Equilibria and Asymptotic stability

2.1 Stability analysis of equilibrium

Let us consider an n-dimensional nonlinear continuous-time dynamical system. For our future purposes we will focus on a particular class of system, as the one given in Su et al. [26]

$$\frac{dx_i(t)}{dt} = \delta_i\left(-x_i(t) + f_i(x(t))\right),\ i = 1,\ldots,n \quad x(0) = x_0 \tag{2.1}$$

where $x_i(t) \in \mathbb{R}_+$ is the i^{th} system state at time t, $f_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ is a continuous function of x and δ_i is a positive constant called *degradation rate*. By defining $\Delta = diag\{\delta_1, \delta_2, \ldots, \delta_n\}$ the problem stated in 2.1 can be rewritten as

$$\dot{x}(t) = \Delta\left(-x(t) + f(x(t))\right), \quad x(0) = x_0 \tag{2.2}$$

where $x = (x_1, x_2, \ldots, x_n)^T$ is a vector in $\mathbb{R}^n_+$ and $f = (f_1, f_2, \ldots, f_n)^T$ is a vector field.
We want to find the conditions under which system (2.2) admits a unique positive equilibrium point. Given the structure of (2.2), these corresponds to the conditions on f such that f admits a unique positive fixed point, i.e.,

$$\exists!\ x^* \in int(\mathbb{R}^n_+),\ \text{such that}\ f(x^*) = x^* \tag{2.3}$$

The following theorem gives us a sufficient condition for the existence of such positive fixed point. Different fixed point theorem can be found in literature [17, 23] but here we focus on Tarski's fixed point theorem.

Theorem 2.1.1 (Tarski fixed point theorem)**.** *Given $\mathscr{D} \subset \mathbb{R}^n_+$ open and convex subset. Assume $f : \mathscr{D} \to \mathscr{D}$ is a non-decreasing vector field such that*

i. $\exists\, a \in \mathscr{D}$, $a > 0$ *such that* $f(a) > a$

ii. $\exists\, b \in \mathscr{D}$, $b > a$ *such that* $f(b) < b$

Then $\exists\, x^* \in \mathscr{D}$, *such that* $f(x^*) = x^*$.

Proof. For proof see Tarski [29]. ■

Under some assumptions on f we will now show that f has a *unique* fixed point providing it satisfies the condition of Theorem 2.1.1. The following result is from Kennan [18], proof is reported for completeness.

Theorem 2.1.2. *Suppose* $f = (f_1, f_2, \dots, f_n)$ *be a vector field from* $\mathbb{R}^n_+$ *to* $\mathbb{R}^n_+$ *such that*

i. *f is increasing*

ii. *f is strictly concave, i.e., f_i strictly concave for* $i = 1, \dots, n$

iii. $f(0) \geq 0$

iv. $\exists\, a \in \mathbb{R}^n_+$, $a > 0$ *such that* $f(a) > a$

v. $\exists\, b \in \mathbb{R}^n_+$, $b > a$ *such that* $f(b) < b$

Then there exists a unique positive vector $x^* \in int(\mathbb{R}^n_+)$, *such that* $f(x^*) = x^*$

Proof. The existence part has already been shown since conditions *iv* and *v* are the conditions for existence from Tarski's Theorem in 2.1.1.
To show uniqueness, suppose $x > 0$ is any fixed point of f. Let $g : \mathbb{R}^n_+ \to \mathbb{R}^n : x \mapsto -x + f(x)$. Note that the strict concavity of f implies that g is strictly concave.
Suppose $y > 0$ and $g(y) \geq 0$. Let

$$\alpha = min\left\{\frac{x_j}{y_j},\, j = 1, \dots, n\right\} = \frac{x_r}{y_r}$$

Then $\alpha > 0$ because $x > 0$ and $y > 0$. If $a \geq 1$ the $y \leq x$. Otherwise let $w = \alpha y$, with $g(w) > 0$ because g is strictly concave, and $g(y) \geq 0$. Then $w \leq x$ and $w_r = x_r$, so $g_r(x) - g_r(w) = f_r(x) - f_r(w) \geq 0$ because f is increasing. But this implies $0 = g_r(x) \geq g_r(w) > 0$, a contradiction. Thus $y > 0$ and $g(y) \geq 0$ implies $y \leq x$.
Now if $y > 0$ is a fixed point of f then, since g(x) = 0, the same argument with the roles of x and y reversed gives $x \leq y$, so $y = x$. ■

We now give some conditions for the existence of a and b as stated in the previous theorems if the function f is increasing and strictly concave. The next proposition will show that a function which is non-decreasing and strictly concave must be increasing.

Proposition 2.1.1. *Let $f = (f_1, f_2, \dots, f_n)$ be a vector field from $\mathbb{R}^n_+$ to $\mathbb{R}^n_+$. If f is strictly concave and non-decreasing then f is increasing.*

Proof. Suppose f is not increasing, i.e, $\exists\, x, y \in \mathbb{R}^n_+$ such that

$$x < y \Rightarrow f_k(x) = f_k(y)$$

for some $k \in \{1, \dots, n\}$. From the strict concavity of f the following inequality holds

$$f_k\left(\alpha x + (1-\alpha)y\right) > \alpha f_k(x) + (1-\alpha) f_k(y), \;\; \forall \alpha \in (0,1)$$

if $z \triangleq \alpha x + (1-\alpha)y$ and since $f_k(x) = f_k(y)$ the inequality becomes

$$f_k(z) > f_k(y) \tag{2.4}$$

to conclude we need to observe that $z < y$. Since $x < y$, i.e. $\alpha x < \alpha y$ for all positive α, we obtain

$$z = \alpha x + (1-\alpha)y < \alpha y + (1-\alpha)y \Rightarrow z < y$$

putting together this last inequality and the one in (2.4) we obtain a contradiction. ■

Before stating and demonstrating the main results of this section, we need the next lemma. Note that the irreducibility of the Jacobian matrix of f means, from Proposition 1.2.4, that the influence graph of $\frac{\partial f}{\partial x}(x)$ is strongly connected, or equivalently from Proposition 1.16 $h = k = 1$, i.e. there is only one isolated subsystem which is equivalent to the whole system.

Lemma 2.1.1. *Let $f = (f_1, f_2, \dots, f_n)$ be a vector field from $\mathbb{R}^n_+$ to $\mathbb{R}^n_+$, strictly concave, increasing and $f(0) = 0$. Let the Jacobian matrix $\frac{\partial f}{\partial x}(x) \geq 0$ be irreducible $\forall x \in int\left(\mathbb{R}^n_+\right)$. If $\exists\, b > 0, b \in int\left(\mathbb{R}^n_+\right)$ such that $f(b) \leq b$ then $\rho\left(\frac{\partial f}{\partial x}(b)\right) < 1$.*

Proof. The lemma states that if $f(b) \leq b$ then

$$\rho\left(\frac{\partial f}{\partial x}(b)\right) < 1.$$

By contradiction let us assume that $\rho\left(\frac{\partial f}{\partial x}(b)\right) \geq 1$. Then by strict concavity of f the following holds

$$f(0) < f(b) + \frac{\partial f}{\partial x}(b)(0-b)$$

from $f(0) = 0$ and $f(b) \leq b$ it follows immediately that

$$0 < b - \frac{\partial f}{\partial x}(b)b. \tag{2.5}$$

Now, let w_0^T be the left Perron-Frobenius positive eigenvector vector corresponding to $\rho\left(\frac{\partial f}{\partial x}(b)\right)$. By multiplying both sides of (2.5) by w_0^T we have

$$0 < w_0^T b - w_0^T \frac{\partial f}{\partial x}(b)b$$

from $w_0^T \frac{\partial f}{\partial x}(b)b = \rho\left(\frac{\partial f}{\partial x}(b)\right) w_0^T b$ and by taking to the left side the second term

$$\rho\left(\frac{\partial f}{\partial x}(b)\right) w_0^T b < w_0^T b$$

which is clearly a contradiction if $\rho\left(\frac{\partial f}{\partial x}(b)\right) \geq 1$. ■

Corollary 2.1.1.1. *Let $f = (f_1, f_2, \ldots, f_n)$ be a vector field from $\mathbb{R}_+^n$ to $\mathbb{R}_+^n$, strictly concave, increasing and $f(0) = 0$. Let the Jacobian matrix $\frac{\partial f}{\partial x}(x) \geq 0$ be irreducible $\forall x \in int\left(\mathbb{R}_+^n\right)$. If $\exists\, x^* > 0,\ x^* \in int\left(\mathbb{R}_+^n\right)$ such that $f(x^*) = x^*$ then $\rho\left(\frac{\partial f}{\partial x}(x^*)\right) < 1$.*

Proof. It is an immediate consequence of Lemma 2.1.1 ■

Remark. *This last corollary states that the spectral radius of the Jacobian of f calculated in positive fixed point x^* for f must be such that $\rho\left(\frac{\partial f}{\partial x}(x^*)\right) < 1$.*

Proposition 2.1.2. *Let $f = (f_1, f_2, \ldots, f_n)$ be a vector field from $\mathbb{R}_+^n$ to $\mathbb{R}_+^n$, strictly concave, increasing and $f(0) = 0$. Let the Jacobian matrix $\frac{\partial f}{\partial x}(x) \geq 0$ be irreducible $\forall x \in int\left(\mathbb{R}_+^n\right)$. Let $g : \mathbb{R}_+^n \to \mathbb{R}^n : x \mapsto -x + f(x)$. If $\exists\, b > 0, b \in \mathbb{R}_+^n$ such that $g(b) \leq 0$ then g is decreasing for all $x > b$.*

Proof. If $g(b) \leq 0$ then $f(b) \leq b$. From the preceding lemma 2.1.1 we have that

$$\rho\left(\frac{\partial f}{\partial x}(b)\right) < 1$$

If w_0^T is the Perron-Frobenius left eigenvector of $\frac{\partial f}{\partial x}(b)$ corresponding to $\rho\left(\frac{\partial f}{\partial x}(b)\right)$ then it is also the left eigenvector of $\frac{\partial g}{\partial x}(b)$ corresponding to $-\varepsilon$, with $\varepsilon > 0$ a real positive scalar (see equations 2.20 and 2.21 for proof), i.e.,

$$w_0^T\left(\frac{\partial g}{\partial x}(b)\right) = -\varepsilon w_0^T.$$

We would like to show that g is decreasing for all $x > b$. We assume there exists a vector x with $x \geq b$, $x \neq b$ and such that $g(x) \geq g(b)$. From strict concavity of g

$$g(x) < g(b) + \frac{\partial g}{\partial x}(b)(x-b).$$

Multiplying both sides by w_0^T we get

$$w_0^T g(x) < w_0^T g(b) + w_0^T \frac{\partial g}{\partial x}(b)(x-b)$$

observing that $w_0^T \frac{\partial g}{\partial x}(b) = -\varepsilon w_0^T$ and taking everything to the left hand side of the inequality, it becomes

$$w_0^T\left(g(x) - g(b) + \varepsilon(x-b)\right) < 0$$

since w_0^T is strictly positive, the inequality becomes

$$g_i(x) - g_i(b) + \varepsilon(x_i - b_i) < 0$$

for some $i \in \{1, \ldots, n\}$, which is

$$g_i(x) < g_i(b) - \varepsilon(x_i - b_i)$$

since $x - b \geq 0$ the inequality yields

$$g_i(x) < g_i(b).$$

This last inequality shows that there is at least one element such that $g_i(x) < g_i(b)$. But we must show that $g_i(x) < g_i(b)$ for all $i = 1, \ldots, n$.
By contradiction let us suppose that there exists $k \in \{1, \ldots, n\}$ such that

$$g_k(x) \geq g_k(b) \tag{2.6}$$

now g_k is strictly concave because it is a positive linear combination of concave functions,

where by assumptions f_k is strictly concave. This means that the upper contour set $S_\alpha = \{x \in \mathbb{R}^n_+ : g_k(x) \geq \alpha\}$ is convex for all $\alpha \in \mathbb{R}$.
By choosing $\alpha = g_k(b) + \varepsilon$, such that $\varepsilon > 0$ and $\alpha < g_k(x)$, it is clear that $b \notin S_\alpha$ while $x \in S_\alpha$.
Let us define $\bar{z}$ as $\bar{z} = \bar{\lambda}x + (1-\bar{\lambda})b$, where $\bar{\lambda}$ is the smallest real number in $(0,1)$ such that $g(\bar{z}) \in S_\alpha$. If $\alpha \leq 0$ then $0 \in S_\alpha$. Then from the strict concavity of g_k the convex combination of 0 and $\bar{z}$ should lie in S_α. But $g(\lambda\bar{z}) < \alpha$ for $\lambda \to 1^-$. This shows that S_α has holes, i.e it is not convex. The case where $\alpha > 0$ is analogous. This means that $g(x) < 0$.
By applying the same reasoning to any $z > x$ we can conclude that $g(z) < g(x)$, and thus that g is decreasing for all $x \geq b$. ■

Corollary 2.1.2.1. *Let $f = (f_1, f_2, \dots, f_n)$ be a vector field from $\mathbb{R}^n_+$ to $\mathbb{R}^n_+$, strictly concave and increasing. If $\exists b \in \mathbb{R}_+$ such that $f(b) < b$ then $f(\bar{b}) < \bar{b}$ $\forall \bar{b} > b$.*

Proof. Immediate consequence of Proposition 2.1.2. ■

Lemma 2.1.2. *Let $f = (f_1, f_2, \dots, f_n)$ be a vector field from $\mathbb{R}^n_+$ to $\mathbb{R}^n_+$, strictly subhomogeneous of degree $0 < \tau \leq 1$ and such that $f(x) \geq 0$ $\forall x \in \mathbb{R}^n_+$. Let the Jacobian matrix $\frac{\partial f}{\partial x}(x) \geq 0$ be irreducible $\forall x \in int\left(\mathbb{R}^n_+\right)$. If $\exists$ $b > 0, b \in int\left(\mathbb{R}^n_+\right)$ such that $f(b) \leq b$ then $\rho\left(\frac{\partial f}{\partial x}(b)\right) < 1$.*

Proof. The lemma states that if $f(b) \leq b$ then

$$\rho\left(\frac{\partial f}{\partial x}(b)\right) < 1.$$

By contradiction let us assume that $\rho\left(\frac{\partial f}{\partial x}(b)\right) \geq 1$. Then by strict subhomogeneity of f Lemma (1.2.1) holds

$$\frac{\partial f}{\partial x}(b)b < \tau f(b)$$

since $\tau \leq 1$ and $f(b) \leq b$ it follows immediately that

$$\frac{\partial f}{\partial x}(b)b < b \tag{2.7}$$

Now, let w_0^T be the left Perron-Frobenius positive eigenvector vector corresponding to $\rho\left(\frac{\partial f}{\partial x}(b)\right)$.
By multiplying both sides of (2.5) by w_0^T we have

$$0 < w_0^T b - w_0^T \frac{\partial f}{\partial x}(b)b$$

from $w_0^T \frac{\partial f}{\partial x}(b)b = \rho\left(\frac{\partial f}{\partial x}(b)\right) w_0^T b$ and by taking to the left side the second term

$$\rho\left(\frac{\partial f}{\partial x}(b)\right) w_0^T b < w_0^T b$$

which is clearly a contradiction if $\rho\left(\frac{\partial f}{\partial x}(b)\right) \geq 1$. ■

Corollary 2.1.2.2. *Let $f = (f_1, f_2, \dots, f_n)$ be a vector field from $\mathbb{R}^n_+$ to $\mathbb{R}^n_+$ strictly subhomogeneous of degree $0 < \tau \leq 1$ and such that $f(x) \geq 0$ $\forall x \in \mathbb{R}^n_+$. Let the Jacobian matrix $\frac{\partial f}{\partial x}(x) \geq 0$ be irreducible $\forall x \in int\left(\mathbb{R}^n_+\right)$. If $\exists\, x^* > 0$, $x^* \in int\left(\mathbb{R}^n_+\right)$ such that $f(x^*) = x^*$ then $\rho\left(\frac{\partial f}{\partial x}(x^*)\right) < 1$.*

Proof. It is an immediate consequence of Lemma 2.1.2 ■

2.2 Existence of equilibrium and asymptotic stability

When dealing with strictly concave and increasing functions we can find some conditions that guarantee the existence of a fixed point.

Theorem 2.2.1 (Spectral radius conditions)**.** *Consider system* (2.2)*. Let $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a differentiable strictly concave, increasing and such that $f(0) = 0$. Let $\frac{\partial f}{\partial x}(x) \geq 0$ $\forall x \in \mathbb{R}^n_+$ be irreducible for all $x \in \mathbb{R}^n_+$. Then*

1. $\rho\left(\frac{\partial f}{\partial x}(0)\right) = \lambda_0 > 1$ *if and only if* $\exists\, a \in \mathbb{R}^n_+$, $a > 0$ *such that* $f(a) > a$
2. $\exists\, \bar{x} \in \mathbb{R}^n_+$ *such that* $\rho\left(\frac{\partial f}{\partial x}(\bar{x})\right) = \zeta_0 < 1$ *if and only if* $\exists\, b \in \mathbb{R}^n_+$, $b > a$ *such that* $f(b) < b$

Proof of 1. Suppose $\rho\left(\frac{\partial f}{\partial x}(0)\right) = \lambda_0 \leq 1$ and there exists $a \in \mathbb{R}^n_+$, $a > 0$ such that $f(a) > a$. We should find a contradiction. Let $w_0^T > 0$ be the left eigenvector of $\frac{\partial f}{\partial x}(0)$ corresponding to λ_0 (see Fornasini [13]). Let $g : \mathbb{R}^n_+ \to \mathbb{R}^n$ be defined as $g(x) = f(x) - x$. From the strict concavity of g and since $f(a) > a \Rightarrow g(a) > 0$, the following relation holds

$$g(a) < g(0) + \left(\frac{\partial g}{\partial x}(0)\right) a = \left(-I + \left(\frac{\partial f}{\partial x}(0)\right)\right) a \tag{2.8}$$

multiplying both sides by w_0^T the inequality becomes

$$w_0^T g(a) < -w_0^T a + w_0^T \left(\frac{\partial f}{\partial x}(0)\right) a = -w_0^T a + \lambda_0 w_0^T a \tag{2.9}$$

taking everything to the left side we obtain

$$w_0^T\left(g(a)+a-\lambda_0 a\right)<0 \tag{2.10}$$

since $w_0^T > 0$ and defining ε as $\varepsilon \triangleq 1-\lambda_0 \geq 0$ implies that there exists $i \in \{1,\ldots,n\}$ such that

$$g_i(a)+(1-\lambda_0)a_i<0 \Rightarrow g_i(a)<-\varepsilon a_i \leq 0 \tag{2.11}$$

this is a contradiction since our hypothesis implies $a_i > 0$ and $g_i(a) > 0$ for all i.
Now suppose $\rho\left(\frac{\partial f}{\partial x}(0)\right) = \lambda_0 > 1$ and we want to show that there exists $a \in \mathbb{R}^n_+$, $a > 0$ such that $f(a) > a$.
From Taylor's theorem [8] and from C^1 assumptions for f we have that for every $i = 1,\ldots,n$ the following holds

$$f_i(x)=f_i(x_0)+\nabla f_i(x_0)(x-x_0)+h_i(x-x_0) \tag{2.12}$$

and $h_i : \mathbb{R}^n \to \mathbb{R}$ is such that

$$\lim_{x\to x_0}\frac{h_i(x-x_0)}{\|x-x_0\|}=0 \tag{2.13}$$

which by joining all equations yields

$$f(x)=f(x_0)+\frac{\partial f}{\partial x}(x_0)(x-x_0)+h(x-x_0). \tag{2.14}$$

Let $v_0 > 0$ be such that $\frac{\partial f}{\partial x}(0)v_0 = \rho\left(\frac{\partial f}{\partial x}(0)\right)v_0$, i.e. v_0 is the positive eigenvector corresponding to the Perron-Frobenius eigenvalue $\rho\left(\frac{\partial f}{\partial x}(0)\right) = \lambda_0 > 1$.
From Taylor's approximation in (2.14) and by choosing $x_0 \equiv 0$ and $x = a$, we have

$$f(a)=f(0)+\frac{\partial f}{\partial x}(0)a+h(a). \tag{2.15}$$

Since we are interested in finding a vector $a > 0$ such that $f(a) > a$, let us choose $a = \gamma v_0$, $\gamma > 0$. The vector a is clearly a positive vector and $\frac{1}{\gamma}h(a) \to 0$ for $\gamma \to 0$. With these choices the last equation becomes

$$f(a)=\rho\left(\frac{\partial f}{\partial x}(0)\right)a+h(a)=\lambda_0 a+h(a) \tag{2.16}$$

by rewriting λ_0 as $\lambda_0 = 1+\varepsilon$, $\varepsilon > 0$, the equation is then

$$f(a) = a + \varepsilon a + h(a). \tag{2.17}$$

By recalling that $a = \gamma v_0$, it is

$$f(a) = a + \gamma\left(\varepsilon v_0 + \frac{1}{\gamma}h(a)\right) \tag{2.18}$$

and since $\frac{1}{\gamma}h(a) \to 0$ for $\gamma \to 0$ we can make $\varepsilon v_0 + (1/\hat{\gamma})h(a) > 0$ for an appropriate small $\hat{\gamma} \neq 0$. The final equation is then

$$f(a) = a + \hat{\gamma}(\text{something positive}) > a, \tag{2.19}$$

this completes the proof of this first part. ■

Proof of 2. Suppose there $\exists\, \bar{x} \in \mathbb{R}^n_+$ such that $\rho\left(\frac{\partial f}{\partial x}(\bar{x})\right) = \zeta_0 < 1$. We first show that there exists $b \in \mathbb{R}^n_+$, $b > a$ such that $f(b) < b$. If we define $g : \mathbb{R}^n_+ \to \mathbb{R}^n$ as $g(x) = f(x) - x$ the following relation holds

$$f(b) < b \Leftrightarrow g(b) < 0$$

we assume there exists $i \in \{1,\dots,n\}$ such that $g_i(\bar{x}) \geq 0$ otherwise we would have concluded since $g(\bar{x}) < 0 \Rightarrow f(\bar{x}) < \bar{x}$. From the definition of spectral radius we have that

$$\zeta_0 = max\left\{|\lambda|, \lambda \in \Lambda\left(\frac{\partial f}{\partial x}(\bar{x})\right)\right\}$$

but since $\frac{\partial f}{\partial x}(\bar{x}) \geq 0$ and also irreducible, from Perron-Frobenius Theorem 1.2.1 we have that $\zeta_0 \in \Lambda\left(\frac{\partial f}{\partial x}(\bar{x})\right)$, i.e. $\zeta_0 \in \mathbb{R}_+$. With the given definition of g its Jacobian matrix in $\bar{x}$ results to be

$$\frac{\partial g}{\partial x}(\bar{x}) = -I + \frac{\partial f}{\partial x}(\bar{x})$$

and we can easily derive its eigenvalue with the maximum real part

$$\begin{aligned} -\varepsilon &\triangleq max\left\{\mathscr{R}e(\lambda), \lambda \in \Lambda\left(\tfrac{\partial g}{\partial x}(\bar{x})\right)\right\} \\ &= max\left\{\mathscr{R}e(\lambda), \lambda \in \Lambda\left(-I + \tfrac{\partial f}{\partial x}(\bar{x})\right)\right\} \\ &= -1 + max\left\{\mathscr{R}e(\lambda), \lambda \in \Lambda\left(\tfrac{\partial f}{\partial x}(\bar{x})\right)\right\} \\ &= -1 + \zeta_0 \end{aligned} \tag{2.20}$$

and of course $\varepsilon > 0$ since $\zeta_0 < 1$. The hypothesis that the Jacobian matrix of f is irreducible implies that ζ_0 is the Perron-Frobenius eigenvalue of $\frac{\partial f}{\partial x}(\bar{x})$ and thus the related eigenvector v is positive, i.e $\mathrm{v} \in \mathbb{R}^n_+, \mathrm{v} > 0$. We now note that the same positive vector v is the eigenvector corresponding to the eigenvalue $-\varepsilon$ for the Jacobian matrix of g. This is clearly shown by the following equation.

$$\left(\frac{\partial g}{\partial x}(\bar{x})\right)\mathrm{v} = \left(-I + \frac{\partial f}{\partial x}(\bar{x})\right)\mathrm{v} = -\mathrm{v} + \frac{\partial f}{\partial x}(\bar{x})\mathrm{v} = (-1+\zeta_0)\mathrm{v} = -\varepsilon \mathrm{v}. \tag{2.21}$$

This part of the proof is concluded if we show that there exists a b such that $\mathrm{g(b)} < 0$. Let us define b as

$$b = \bar{x} + \gamma v \quad \gamma \in \mathbb{R}_+, \gamma > 0 \tag{2.22}$$

since $v > 0$ and γ is positive it is clear that $b > \bar{x}$. The vector field g is strictly concave since it is a positive linear combination of concave functions, with f strictly concave. Then $\forall x, y \in \mathbb{R}^n_+, x \neq y$ the following holds

$$g(y) < g(x) + \frac{\partial g}{\partial x}(x)(y-x)$$

then choosing $y \equiv b$ and $x \equiv \bar{x}$ the relations becomes

$$g(\mathrm{b}) < g(\bar{\mathrm{x}}) + \frac{\partial g}{\partial x}(\bar{\mathrm{x}})(\mathrm{b} - \bar{\mathrm{x}}) \tag{2.23}$$

from definition (2.22), $\mathrm{b} - \bar{\mathrm{x}} = \gamma \mathrm{v}$ together with (2.21) the previous relation becomes

$$g(\mathrm{b}) < g(\bar{\mathrm{x}}) - \varepsilon\gamma\mathrm{v} \tag{2.24}$$

by choosing an opportune γ the right hand side can be made negative. For example

$$\gamma \triangleq \frac{1}{\varepsilon}\frac{max_i\left\{g_i(\bar{\mathrm{x}}), i = 1, \ldots, n\right\}}{min_j\left\{\mathrm{v}_j, j = 1, \ldots, n\right\}} \tag{2.25}$$

implies

$$g(\mathrm{b}) < g(\bar{\mathrm{x}}) - \frac{max_i\left\{g_i(\bar{\mathrm{x}}), i = 1, \ldots, n\right\}}{min_j\left\{\mathrm{v}_j, j = 1, \ldots, n\right\}}\mathrm{v} \leq g(\bar{\mathrm{x}}) - max_i\left\{g_i(\bar{\mathrm{x}}), i = 1, \ldots, n\right\}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} \leq 0 \tag{2.26}$$

since $\left(\mathrm{v}_i / min_j\left\{\mathrm{v}_j, j = 1, \ldots, n\right\}\right) \geq 1, \forall i$. The proof is concluded since

$$g(\mathrm{b}) < 0 \Rightarrow \mathrm{f(b)} < \mathrm{b} \tag{2.27}$$

To conclude that $b > a$, we recall from the proof of part 1 that a can be chosen arbitrarily small and proportional to the Perron-Frobenius eigenvector in 0.

To show the necessity part, assume that $\exists\ a \in \mathbb{R}^n_+,\ a > 0$ such that $f(a) > a$ and there $\exists\ b \in \mathbb{R}^n_+,\ b > a$ such that $f(b) < b$. We also assume that $\nexists\ \bar{x} > 0$ such that $\rho\left(\frac{\partial f}{\partial x}(\bar{x})\right) = \zeta_0 < 1$, i.e $\rho\left(\frac{\partial f}{\partial x}(x)\right) \geq 1,\ \forall x \in \mathbb{R}^n_+$. Define $\bar{b}$ as

$$\bar{b} = a + \gamma v \quad \gamma \in \mathbb{R}_+, \gamma > 0$$

where v is the Perron-Frobenius eigenvector of $\frac{\partial f}{\partial x}(x)$, thus positive and γ is an opportune positive real constant such that $\bar{\mathrm{b}} > \mathrm{b}$. From Proposition 2.1.2 $f(b) < b$ implies that

$$f(\bar{b}) < \bar{b}$$

then $\forall x, y \in \mathbb{R}^n_+, x \neq y$ the following relation holds since f is strictly concave

$$f(y) < f(x) + \frac{\partial f}{\partial x}(x)(y - x)$$

choosing $y \equiv a$ and $x \equiv \bar{\mathrm{b}}$ the relations becomes

$$f(a) < f(\bar{b}) - \frac{\partial f}{\partial x}(\bar{b})(\bar{b} - a) \tag{2.28}$$

since $\rho\left(\frac{\partial f}{\partial x}(\bar{b})\right) \geq 1$ by assumption and $\bar{b} - a = \gamma v$ the next equation holds

$$\frac{\partial f}{\partial x}(\bar{b})(\bar{b} - a) = \gamma \frac{\partial f}{\partial x}(\bar{b})\mathrm{v} = \gamma\rho\left(\frac{\partial f}{\partial x}(\bar{b}))\right)\mathrm{v} = \rho\left(\frac{\partial f}{\partial x}(\bar{b}))\right)(\bar{b} - a)$$

since $\rho\left(\frac{\partial f}{\partial x}(\bar{b})\right) \geq 1$ is the Perron-Frobenius eigenvalue. The inequality in (2.28) becomes

$$f(a) < f(\bar{b}) - \rho\left(\frac{\partial f}{\partial x}(\bar{b})\right)(\bar{b} - a) < f(\bar{b}) - (\bar{b} - a) \tag{2.29}$$

from $f(\bar{b}) - \bar{b} < 0$ we obtain

$$f(a) < \left(f(\bar{b}) - \bar{b}\right) + a < a \tag{2.30}$$

which is a contradiction. ■

Theorem 2.2.2. *Consider system (2.2). Let $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a differentiable strictly concave, increasing and such that $f(0) = 0$. Let $\frac{\partial f}{\partial x}(x) \geq 0$ $\forall x \in \mathbb{R}^n_+$ be irreducible for all $x \in \mathbb{R}^n_+$. Then*

1. *if $\rho\left(\frac{\partial f}{\partial x}(0)\right) = \sigma_0 < 1$ then the origin of system (2.2) is asymptotically stable and the domain of attraction $A(0)$ includes $\mathbb{R}^n_+$, i.e. $\mathbb{R}^n_+ \subset A(0)$.*

2. *if $\rho\left(\frac{\partial f}{\partial x}(0)\right) = \lambda_0 > 1$ and there $\exists \bar{x} \in \mathbb{R}^n_+$ such that $\rho\left(\frac{\partial f}{\partial x}(\bar{x})\right) = \zeta_0 < 1$ then the system (2.2) admits a unique positive equilibrium point x^* which is asymptotically stable and its domain of attraction is such that $\mathbb{R}^n_+ \backslash \{0\} \subset A(x^*)$.*

Proof of 1. From the Perron-Frobenius theorem and from the irreducibility of the Jacobian matrix of f in zero, there exists a positive vector w_0^T such that w_0^T is the left eigenvector corresponding to the eigenvalue σ_0 of $\frac{\partial f}{\partial x}(0)$.
For the system (2.2), the diagonal matrix Δ is positive definite which implies that Δ^{-1} is positive definite. Then let $V : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a Lyapunov's function (see Khalil and Grizzle [19]) such that

$$V = \frac{1}{2} x^T \left(\Delta^{-1} w_0 w_0^T \Delta^{-1}\right) x \tag{2.31}$$

where the matrix $\left(\Delta^{-1} w_0 w_0^T \Delta^{-1}\right)$ is clearly symmetric and strictly positive because $w_0^T > 0$ and $\delta_i > 0$ for $i = 1, \ldots, n$. This implies that V is strictly positive $\forall\, x \in \mathbb{R}^n_+ \backslash \{0\}$. From the differentiability of V, we have

$$\begin{aligned} \dot{V} &= x^T \Delta^{-1} w_0 w_0^T \Delta^{-1} \dot{x} \\ &= x^T \Delta^{-1} w_0 w_0^T \left(-x + f(x)\right) \end{aligned} \tag{2.32}$$

Now from strict concavity of f, for every $x, y \in \mathbb{R}^n_+, x \neq y$ the following holds

$$f(x) < f(y) + \frac{\partial f}{\partial x}(y)(x - y)$$

with $y = 0$ it is

$$f(x) < \frac{\partial f}{\partial x}(0) x$$

by multiplying both sides by w_0^T we obtain

$$w_0^T f(x) < \rho\left(\frac{\partial f}{\partial x}(0)\right) w_0^T x = \sigma_0 w_0^T x$$

the assumption $\sigma_0 < 1$ yields the *diagonal dominance* condition

$$w_0^T f(x) < w_0^T x. \tag{2.33}$$

Going back to the Lyapunov's function in (2.32) we have that

$$\begin{aligned} \dot{V} &= -x^T \Delta^{-1} w_0 w_0^T x + x^T \Delta^{-1} w_0 w_0^T f(x) \\ &= x^T \Delta^{-1} w_0 \left(-w_0^T x + w_0^T f(x)\right) \end{aligned}$$

from condition (2.33) it is clear that $-w_0^T x + w_0^T f(x) < 0$. This implies $\dot{V} < 0$ for all $x \in \mathbb{R}^n_+ \backslash \{0\}$ since $x^T \Delta^{-1} w_0 > 0$. The proof holds globally since V is radially unbounded, which implies that $x = 0$ is globally asymptotically stable.

■

Proof of 2. From condition 2, Theorem 2.2.1 holds. So there exists a unique positive fixed point, i.e.

$$\exists!\ x^* \in int(\mathbb{R}^n_+), \text{ such that } f(x^*) = x^*.$$

With the following change of variable $y = x - x^*$ and recalling that $x \in \mathbb{R}^n_+$ we have that

$$\begin{aligned} \dot{y} &= \dot{x} \\ y &\geq -x^* \end{aligned}$$

And the fixed point $x = x^*$ is now equivalent to $y = 0$. By choosing a Lyapunov function as follows

$$V = \frac{1}{2} y^T \Delta^{-1} y$$

and by defining g as $g : \mathbb{R}^n_+ \to \mathbb{R}^n : x \mapsto -x + f(x)$, we obtain

$$\begin{aligned} \dot{V} &= y^T \Delta^{-1} \dot{y} \\ &= y^T \left(-(y + x^*) + f(y + x^*)\right) \\ &= y^T g(y + x^*) \\ &= \sum_{i=1}^{n} y_i g_i(y + x^*) \end{aligned}$$

note that $g_i(y + x^*) \equiv g_i(x)$. We have two cases to analyse.

a. If $-x^* \leq y \leq 0$ and $\exists k$ such that $y_k < 0$, we have $g(x) \geq 0$ and $g_k(x) > 0$. In fact from proposition 2.1.2, g is strictly decreasing beyond a point x where $g_i(x) < 0$, that is $g_i(\bar{x}) < 0\ \forall \bar{x} \geq x$. Since a positive fixed point exists beyond x, it follows that is it can not be $g_k(x) \leq 0$, or, more directly, from Proposition 2.1.2 we have that if $y < 0$ then

$g(x) > 0$.
From strict concavity of g_k, we already know that $g_k(x^*) = 0$. So from $y_k < 0$, which implies $x \neq x^*$, if $g_k(x) = 0$ we can conclude that g_k is not strictly concave. So must be $g_k(x) > 0$. This implies that $y_i g_i(x) \leq 0 \, \forall i$, and in particular $y_k g_k(x) < 0$.

b. Analogously, if $y \geq 0$ and $\exists k$ such that $y_k > 0$, we have $g_i(x) \leq 0$ while $g_k(x) < 0$. Or again, from Proposition 2.1.2 if $y > 0$ then $g(x) < 0$.

Before concluding we should observe that the two sets $\Omega_1 = \{x \leq x^*\}$ and $\Omega_2 = \{x \geq x^*\}$ are invariant. In fact under our assumptions on f, system (2.2) is cooperative, therefore monotone (see Bokharaie [4], Smith [24]), that is

$$x_0 \leq y_0 \Rightarrow x(t, x_0) \leq x(t, y_0). \tag{2.34}$$

This implies that on Ω_1 we have

$$x_0 \leq x^* \Rightarrow x(t, x_0) \leq x(t, x^*) = x^* \tag{2.35}$$

and for Ω_2,

$$x^* \leq x_0 \Rightarrow x^* = x(t, x^*) \leq x(t, x_0), \;\; \forall t \geq 0. \tag{2.36}$$

Putting together these conditions, we can say that there always exists a strict negative term in the summation of $\dot{V}$ while the other terms are less or equal to zero. We can thus conclude that

$$\dot{V} < 0 \, \forall y \geq -x^* \, , y \neq x^*$$

or equivalently

$$\dot{V} < 0 \, \forall x \in \mathbb{R}^n_+ \backslash \{0\}$$

whenever $x_0 \in \Omega_1 \cup \Omega_2$.
We are left with the case in which $x_0 \notin \Omega_1 \cup \Omega_2$, i.e., x_0 has some elements $x_i \geq x_i^*$ and others $x_j < x_j^*$. In this case there exists two positive real constants $\alpha < 1 \, , \beta > 1$ such that $\alpha x_0 \in \Omega_1$ and $\beta x_0 \in \Omega_2$, it follows immediately

$$\alpha x_0 \leq x_0 \leq \beta x_0 \Rightarrow x(t, \alpha x_0) \leq x(t, x_0) \leq x(t, \beta x_0), \;\; \forall t \geq 0.$$

We already know that $x(t, \alpha x_0) \to x^*$ and $x(t, \beta x_0) \to x^*$ then it must be

$$x(t, x_0) \to x^*.$$

The proof holds globally since V is radially unbounded, this implies that x^* is globally asymptotically stable. ■

2.2.1 System with different degradation rate

If we formulate the problem in 2.1 differently as

$$\frac{dx_i(t)}{dt} = -\delta_i x_i(t) + h_i(\mathrm{x}(t)),\ i = 1,\dots,n \quad x(0) = x_0 \tag{2.37}$$

where $x_i(t) \in \mathbb{R}_+$ is the i^{th} state at time t, $h_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ is the function modelling the influence of all other states to state i and δ_i a positive degradation rate. By defining $\Delta = diag\{\delta_1,\dots,\delta_n\}$ the problem stated in 2.37 can be rewritten as

$$\dot{x}(t) = -\Delta x(t) + h(x(t)), \quad x(0) = x_0 \tag{2.38}$$

where $\mathrm{x} = (x_1, x_2, \dots, x_n)^T$ and $\mathrm{h} = (h_1, h_2, \dots, h_n)^T$ are two vectors in $\mathbb{R}^n_+$. We can make it resemble the form of 2.2, i.e,

$$\dot{\mathrm{x}}(t) = \Delta\left(-x(t) + \Delta^{-1}h(x(t))\right), \quad x(0) = x_0. \tag{2.39}$$

By defining f as $f = \Delta^{-1}h(x(t))$, we can apply Theorem 2.2.2 as is.
We would like to translate the condition

$$\rho\left(\frac{\partial f}{\partial x}(0)\right) > 1 \tag{2.40}$$

and

$$\rho\left(\frac{\partial f}{\partial x}(\bar{x})\right) < 1 \text{ for some } \bar{x} \in \mathbb{R}^n_+ \tag{2.41}$$

directly to the function h.
Firstly, let us find the equivalent of condition (2.40) for the vector field h. Let v_0 be the Perron-Frobenius right eigenvector corresponding to the eigenvalue $\rho\left(\frac{\partial f}{\partial x}(0)\right)$ and w_0^T be the Perron-Frobenius left eigenvector corresponding to the eigenvalue $\rho\left(\frac{\partial h}{\partial x}(0)\right)$, i.e.

$$\begin{aligned} \frac{\partial f}{\partial x}(0)v_0 &= \rho\left(\frac{\partial f}{\partial x}(0)\right)v_0, \\ w_0^T\frac{\partial h}{\partial x}(0) &= w_0^T\rho\left(\frac{\partial h}{\partial x}(0)\right). \end{aligned} \tag{2.42}$$

Since the Jacobian matrix of f is irreducible $\forall x \in \mathbb{R}^n_+$, $\rho\left(\frac{\partial f}{\partial x}(0)\right)$ and $\rho\left(\frac{\partial f}{\partial x}(\bar{x})\right)$ are positive real eigenvalues with positive Perron-Frobenius eigenvectors $v_0 > 0$ and $w_0 > 0$.
From the definition of f the following equivalences holds:

$$\frac{\partial f}{\partial x}(0) = \Delta^{-1}\frac{\partial h}{\partial x}(0) \;\Rightarrow\; \frac{\partial f}{\partial x}(0)v_0 = \Delta^{-1}\frac{\partial h}{\partial x}(0)v_0 \;\Rightarrow\; \rho\left(\frac{\partial f}{\partial x}(0)\right)v_0 = \Delta^{-1}\frac{\partial h}{\partial x}(0)v_0$$

finally yielding

$$\Delta\rho\left(\frac{\partial f}{\partial x}(0)\right)v_0 = \frac{\partial h}{\partial x}(0)v_0. \tag{2.43}$$

Multiplying both sides of (2.43) by w_0^T we obtain

$$\rho\left(\frac{\partial f}{\partial x}(0)\right)w_0^T\Delta v_0 = w_0^T\frac{\partial h}{\partial x}(0)v_0$$

which by the assumptions in (2.42) yields

$$\rho\left(\frac{\partial f}{\partial x}(0)\right)w_0^T\Delta v_0 = \rho\left(\frac{\partial h}{\partial x}(0)\right)w_0^T v_0$$

from $v_0 > 0$, $w_0^T > 0$ we have $w_0^T v_0 > 0$ and $w_0^T\Delta v_0 > 0$. Therefore

$$\rho\left(\frac{\partial f}{\partial x}(0)\right) = \rho\left(\frac{\partial h}{\partial x}(0)\right)\frac{w_0^T v_0}{w_0^T\Delta v_0}. \tag{2.44}$$

Condition (2.40) is then equivalent to

$$\rho\left(\frac{\partial h}{\partial x}(0)\right) > \frac{w_0^T\Delta v_0}{w_0^T v_0}. \tag{2.45}$$

Analogously condition (2.41) is equivalent to

$$\rho\left(\frac{\partial h}{\partial x}(\bar{x})\right) < \frac{\bar{w}_0^T\Delta\bar{v}_0}{\bar{w}_0^T\bar{v}_0}, \tag{2.46}$$

where $\bar{w}_0^T$ and $\bar{v}_0$ are respectively the left and right Perron-Frobenius eigenvectors corresponding to the eigenvalues $\rho\left(\frac{\partial h}{\partial x}(\bar{x})\right)$ and $\rho\left(\frac{\partial f}{\partial x}(\bar{x})\right)$.
The new conditions in (2.45) and (2.46) are difficult to satisfy because the four eigenvectors depends on the Jacobian matrix of f and h. We would like to find other conditions that do not have the kind of intrinsic dependences. To do so, let us consider the term $\frac{w_0^T\Delta v_0}{w_0^T v_0}$, which

expanded yields

$$\frac{w_0^T \Delta v_0}{w_0^T v_0} = \frac{1}{w_0^T v_0} \sum_{i=1}^{n} \delta_i w_{0,i} v_{0,i} \tag{2.47}$$

multiplying the right side by $\delta_{max}/\delta_{max}$ we obtain

$$\frac{w_0^T \Delta v_0}{w_0^T v_0} = \frac{\delta_{max}}{w_0^T v_0} \sum_{i=1}^{n} \frac{\delta_i}{\delta_{max}} w_{0,i} v_{0,i} \tag{2.48}$$

clearly $\frac{\delta_i}{\delta_{max}} \leq 1$ for $i = 1, \ldots, n$ since δ_{max} is the maximum degradation rate. Thus equation (2.48) can then be bounded as follows:

$$\frac{w_0^T \Delta v_0}{w_0^T v_0} = \frac{\delta_{max}}{w_0^T v_0} \sum_{i=1}^{n} \frac{\delta_i}{\delta_{max}} w_{0,i} v_{0,i} \leq \frac{\delta_{max}}{w_0^T v_0} \sum_{i=1}^{n} w_{0,i} v_{0,i} = \frac{\delta_{max}}{w_0^T v_0} w_0^T v_0 \tag{2.49}$$

which finally yields

$$\frac{w_0^T \Delta v_0}{w_0^T v_0} \leq \delta_{max}. \tag{2.50}$$

Therefore the following condition

$$\rho\left(\frac{\partial h}{\partial x}(0)\right) > \delta_{max} \tag{2.51}$$

is a sufficient to guarantee

$$\rho\left(\frac{\partial h}{\partial x}(0)\right) > \frac{w_0^T \Delta v_0}{w_0^T v_0}$$

as stated in (2.45).

Analogously, for the second condition (2.41) let us consider the term $\frac{\bar{w}_0^T \Delta \bar{v}_0}{\bar{w}_0^T \bar{v}_0}$, which expanded yields

$$\frac{\bar{w}_0^T \Delta \bar{v}_0}{\bar{w}_0^T \bar{v}_0} = \frac{1}{\bar{w}_0^T \bar{v}_0} \sum_{i=1}^{n} \delta_i \bar{w}_{0,i} \bar{v}_{0,i} \tag{2.52}$$

multiplying the right side by $\delta_{min}/\delta_{min}$ we obtain

$$\frac{\bar{w}_0^T \Delta \bar{v}_0}{\bar{w}_0^T \bar{v}_0} = \frac{\delta_{min}}{\bar{w}_0^T \bar{v}_0} \sum_{i=1}^{n} \frac{\delta_i}{\delta_{min}} \bar{w}_{0,i} \bar{v}_{0,i} \tag{2.53}$$

clearly $\frac{\delta_i}{\delta_{min}} \geq 1$ for $i = 1, \ldots, n$ since δ_{min} is the minimum degradation rate. Thus equation (2.53) is then

$$\frac{\bar{w}_0^T \Delta \bar{v}_0}{\bar{w}_0^T \bar{v}_0} = \frac{\delta_{min}}{\bar{w}_0^T \bar{v}_0} \sum_{i=1}^{n} \frac{\delta_i}{\delta_{min}} \bar{w}_{0,i} \bar{v}_{0,i} \geq \frac{\delta_{min}}{\bar{w}_0^T \bar{v}_0} \sum_{i=1}^{n} \bar{w}_{0,i} \bar{v}_{0,i} = \frac{\delta_{min}}{\bar{w}_0^T \bar{v}_0} \bar{w}_0^T \bar{v}_0 \tag{2.54}$$

which finally yields

$$\frac{w_0^T \Delta v_0}{w_0^T v_0} \geq \delta_{min}. \tag{2.55}$$

This means that the following condition

$$\rho\left(\frac{\partial h}{\partial x}(\bar{x})\right) < \delta_{min} \tag{2.56}$$

is a sufficient to guarantee

$$\rho\left(\frac{\partial h}{\partial x}(\bar{x})\right) < \frac{\bar{w}_0^T \Delta \bar{v}_0}{\bar{w}_0^T \bar{v}_0}$$

as stated in (2.46).

Putting together condition (2.51) and condition (2.56), Theorem 2.2.2 can then be re-stated as

Theorem 2.2.3. *Consider system (2.39). Let $h : \mathbb{R}_+^n \to \mathbb{R}_+^n$ be a C^1 strictly concave, increasing and such that $h(0) = 0$. Let $\frac{\partial f}{\partial x}(x) \geq 0$ $\forall x \in \mathbb{R}_+^n$ be irreducible for all $x \in \mathbb{R}_+^n$. Then*

1. *if $\rho\left(\frac{\partial h}{\partial x}(0)\right) = \sigma_0 < \delta_{min}$ then the origin of system (2.39) is asymptotically stable and the domain of attraction $A(0)$ includes $\mathbb{R}_+^n$, i.e. $\mathbb{R}_+^n \subset A(0)$.*

2. *if $\rho\left(\frac{\partial h}{\partial x}(0)\right) = \lambda_0 > \delta_{max}$ and there $\exists \bar{x} \in \mathbb{R}_+^n$ such that $\rho\left(\frac{\partial h}{\partial x}(\bar{x})\right) = \zeta_0 < \delta_{min}$ then the system (2.39) admits a unique positive equilibrium point x^* which is asymptotically stable and its domain of attraction is such that $\mathbb{R}_+^n \backslash \{0\} \subset A(x^*)$.*

Proof. See proof of Theorem 2.2.2. ■

2.2.2 Stability on the interior of $\mathbb{R}_+^n$

From theorem 2.2.3 we observe that the domain of attraction $A(x^*)$ of the positive fixed point contains $\mathbb{R}_+^n \backslash \{0\}$. The reason is because we have assumed the vector field f to be increasing, strictly concave and irreducibility of $\frac{\partial f}{\partial x}(x)$ for every $x \in \mathbb{R}_+^n$. We would like to know what happens if these conditions are satisfied only in the interior of $\mathbb{R}_+^n$.

Example 2.2.1. *Let us consider the following example in $\mathbb{R}^4_+$.*

$$\begin{cases} \dot{x}_1 = -\delta_1 x_1 + \sqrt[4]{x_2 x_3 x_4} \\ \dot{x}_2 = -\delta_2 x_2 + \sqrt[4]{x_1 x_3 x_4} \\ \dot{x}_3 = -\delta_3 x_3 + \sqrt[4]{x_1 x_2 x_4} \\ \dot{x}_4 = -\delta_4 x_4 + \sqrt[4]{x_1 x_2 x_3} \end{cases} \tag{2.57}$$

whenever $x_0 \in bd\left(\mathbb{R}^4_+\right)$ and x_0 has at least two zero element then $x(t,x_0) \to 0$. Thus for this example the domain of attraction $A(x^*)$ of the positive fixed point do no longer contain $\mathbb{R}^n_+ \backslash \{0\}$, as stated in theorem 2.2.3, and now contains only $int(\mathbb{R}^n_+)$. The reasons are: $i)$ f is not strictly concave and increasing on the border and $ii)$ the Jacobian of f is not well defined.

We would now like to characterize the points at the border $x_0 \in bd\left(\mathbb{R}^n_+\right)$ such that $x(t,x_0) \to x^*$, where x^* is the positive fixed point of equilibrium (a similar condition can be found in De Leenheer and Aeyels [9]). Under the condition that f_i does not depend on x_i $\forall i$, we can give the following lemma.

Lemma 2.2.1. *Consider system (2.39). Let $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a C^1 strictly concave, increasing $\forall x \in int(\mathbb{R}^n_+)$ and such that $f(0) = 0$. Let $\frac{\partial f}{\partial x}(x) \geq 0$ $\forall x \in \mathbb{R}^n_+$ be irreducible for all $x \in \mathbb{R}^n_+$. Let x^* be the positive equilibrium under condition (2) of Theorem 2.2.3. Let $x_0 \in bd\left(\mathbb{R}^n_+\right)$ be the initial condition. If $f(x_0)$ is such that*

1. *$f(x_0)$ has at least $(n-1)$ non-zero elements then $x(t,x_0) \to x^*$*
2. *$f(x_0)$ has at least $(n-2)$ zero elements then convergence of $x(t,x_0)$ to x^* can not be guaranteed.*

Proof of 1. Suppose for simplicity that $f_i(x_0) \neq 0$ $\forall i = \{2,\ldots,n\}$ and $f_1(x_0) = 0$. Notice that every equation is of the form

$$\dot{x}_i = -\delta_i x_i + f_i(x).$$

So, if $x_i = 0$ then $\dot{x}_i > 0$ $\forall i = \{2,\ldots,n\}$ i.e. x_i can be assumed to be positive. The explicit equation for x_1 is

$$\dot{x}_1 = -\delta_1 x_1 + f_1(x_2,\ldots,x_n)$$

so x_1 becomes positive. This means that the solution $x(t,x_0)$ is now in the interior of $\mathbb{R}^n_+$ ∎

Proof of 2. Suppose for simplicity that $f_i(x_0) \neq 0 \ \forall i = \{3,\dots,n\}$ and $f_1(x_0) = f_2(x_0) = 0$. We also assume $x_1 = x_2 = 0$.

$$\begin{cases} \dot{x}_1 = -\delta_1 x_1 + (x_2)^{\frac{1}{k}} f_1(x) \\ \dot{x}_2 = -\delta_2 x_2 + (x_1)^{\frac{1}{k}} f_2(x) \\ \dot{x}_3 = -\delta_3 x_3 + f_3(x) \\ \vdots \\ \dot{x}_n = -\delta_n x_n + f_n(x) \end{cases} \tag{2.58}$$

with an appropriate choice of k such that $(x_2)^{\frac{1}{k}} f_1(x)$ and $(x_1)^{\frac{1}{k}} f_2(x)$ are strictly concave. In this particular example if the initial condition $x_0 = (x_1, x_2, \dots, x_n)$ is such that $x_1 = x_2 = 0$ then $x_1 = x_2 = 0$ for all future time. Hence the evolution cannot approach x^*. ■

Remark. *Assume f to be C_1, strictly concave and increasing $\forall x \in \mathbb{R}^n_+$. If in addition $\frac{\partial f}{\partial x}(x)$ is irreducible for all $x \in \mathbb{R}^n_+$ then condition 1 of previous lemma can be weakened. In fact if $x_0 \in bd\left(\mathbb{R}^n_+\right)$ such that $f(x_0)$ has at least one non-zero element convergence to x^* is guaranteed. The proof is implicitly included in the proof of Theorem 2.2.2*

2.3 Comparison with Contractive Interference Functions

When dealing with non-negative systems one of the application field is power control in wireless network, see Charalambous [7], Feyzmahdavian et al. [11, 12], Möller and Jönsson [22], Sung and Leung [28], Yates [30]. In Yates [30] the definition of *standard interference functions* was given.

Definition 2.3.1. *A function $I : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is called a standard interference function if for all $p \in \mathbb{R}^n_+$ the following properties are satisfied:*

i Positivity: $I(p) > 0$.

ii Monotonicity: if $p \geq \bar{p}$, then $I(p) \geq I(\bar{p})$

iii Scalability: For all $\alpha > 1$, $\alpha I(p) > I(\alpha p)$

In wireless networks, the positivity property of an interference function is justified by the presence of a non-zero background noise. The scalability property implies that if $p_j \geq I_j(p)$ then $\alpha p_j \geq \alpha I_j(p) > I_j(\alpha p)$ for $\alpha > 1$. That is, if user j has an acceptable connection under power vector p, then user j will have a more than acceptable connection when all powers

are scaled up uniformly.
Not all interference functions can be analysed under this framework. For example, to achieve maximum throughput in wireless networks transmission power should be increased when the interference is low, and the information transmission rate should be adjusted accordingly. This approach is called opportunistic power control (see Leung and Sung [21], [27]). In [27] *generalized interference functions* where introduced

Definition 2.3.2. *A function $I : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is called a two-sided scalable interference function if for all $p \in \mathbb{R}^n_+$, the following properties are satisfied:*

i Positivity: $I(p) > 0$.

ii Two-sided scalability: For all $\alpha > 1$,

$$\frac{1}{\alpha}p \le \bar{p} \le \alpha p \Rightarrow \frac{1}{\alpha}I(p) < I(\bar{p}) < \alpha I(p)$$

It is also shown that standard interference functions are also two-sided scalable.
For our purposes let us consider system (2.1), with $f \equiv I$. In Feyzmahdavian et al. [11] it is shown that if a two-sided scalable function has a fixed point then the system (2.1) is asymptotically stable.

Theorem 2.3.1 (Nominal Power Control Algorithm)**.** *If a two-sided scalable interference function $I : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ as a fixed point $p^* > 0$, then the continuous-time power control algorithm (2.1) is asymptotically stable for any initial condition $p_0 > 0$, and any proportionality constant δ_i.*

However, two-sided scalable interference functions do not guarantee the existence of a positive fixed point and this has to be investigated separately. In [12] a modification of standard interference functions which guarantees the existence of a fixed point was introduced.

Definition 2.3.3. *A function $I : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is said to be c-contractive interference function if for all $p \in \mathbb{R}^n_+$, the following properties are satisfied:*

i Positivity: $I(p) > 0$.

ii Monotonicity: if $p \ge \bar{p}$, then $I(p) \ge I(\bar{p})$

iii Contractivity: There exists a constant $c \in [0,1)$, and a vector $v > 0$ such that for all $\varepsilon > 0$

$$I(p + \varepsilon v) \le I(p) + c\varepsilon v \tag{2.59}$$

Contractive interference functions define contraction mappings in the weighted norm l_∞, which implies that $I(p) = p$ always admits a unique solution in $int(\mathbb{R}^n_+)$.

Proposition 2.3.1. *If a function $I : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is a c-contractive interference function, then it has a unique fixed point $p^* \in int(\mathbb{R}^n_+)$ and*

$$\|I(p) - I(\bar{p})\|^v_\infty \leq c\|p - \bar{p}\|^v_\infty$$

In Feyzmahdavian et al. [11] a characterization of the rate of convergence is given:

Theorem 2.3.2 (Nominal Power Control Algorithm)**.** *If an interference function $I : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is c-contractive, then the continuous-time power control algorithm (2.1) is exponentially stable for any initial condition $p_0 > 0$, and any proportionality constant δ_i. In particular, the solution $p(t)$ of (2.1) satisfies*

$$\|p(t) - p^*\|^v_\infty \leq \|p_0 - p^*\|^v_\infty e^{-\delta_{min}(1-c)t}, \ t \geq 0.$$

In Feyzmahdavian et al. [12] it is shown that many practical interference functions and all the examples in Yates [30] are *c-contractive*.

Now we would like to derive the spectral properties of a c-contractive function.

Proposition 2.3.2. *Let $I : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be c-contractive interference function. Consider system (2.1). If I is differentiable at p then $\rho\left(\frac{\partial I}{\partial p}(p)\right) < 1$*

Proof. From the monotonicity condition of c-contractive interference functions, i.e.

$$p \geq \bar{p} \Rightarrow I_k(p) \geq I_k(\bar{p}), \text{ for all } k = 1, \ldots, n$$

we have that the following holds

$$\nabla I_k(p) \geq 0 \ \ \forall k \Rightarrow \frac{\partial I}{\partial p}(p) = \begin{pmatrix} \nabla I_1(p) \\ \vdots \\ \nabla I_n(p) \end{pmatrix} \geq 0. \tag{2.60}$$

Furthermore, from the contractivity condition given by (2.59), $\varepsilon > 0$ can be chosen arbitrarily. In particular we can make the following choice

$$\varepsilon = \gamma \frac{1}{\|v\|}.$$

Then the inequality in (2.59) holds for every component of I and for every $\gamma > 0$, i.e.

$$I_k(p+\varepsilon v) \leq I_k(p) + c\varepsilon v_k$$

with our choice of ε it becomes

$$I_k\left(p+\gamma\frac{v}{\|v\|}\right) \leq I_k(p) + c\gamma\frac{v_k}{\|v\|}$$

rearranging the inequality and defining $u := v/\|v\|$ as a unit vector (note that $u > 0$ since $v > 0$ by assumptions) we have

$$\frac{I_k(p+\gamma u) - I_k(p)}{\gamma} \leq cu_k$$

from the differentiability of I_k in p and taking the limit for $\gamma \to 0$ we obtain the following (Dal Passo et al. [8])

$$\lim_{\gamma\to 0}\frac{I_k(p+\gamma u) - I_k(p)}{\gamma} = \nabla I_k(p)\cdot u,$$

which is the directional derivative along u. The contractivity condition is then

$$\nabla I_k(p)\cdot u \leq cu_k$$

the previous inequality holds for all $k = 1,\ldots,n$ so it yields

$$\left(\frac{\partial I}{\partial p}(p)\right)u \leq cu. \tag{2.61}$$

Since the Jacobian matrix is non-negative $\forall p \in \mathbb{R}^n_+$ this implies that

$$\rho\left(\frac{\partial I}{\partial p}(p)\right) \geq 0$$

and there exists a non-zero eigenvector $w_0 \in \mathbb{R}^n_+$, $w_0 \neq 0$ such that

$$w_0^T\left(\frac{\partial I}{\partial p}(p)\right) = \rho\left(\frac{\partial I}{\partial p}(p)\right)w_0^T,$$

so from this last equation together with the inequality in (2.61) we have

$$\rho\left(\frac{\partial I}{\partial p}(p)\right)w_0^T u \leq cw_0^T u \tag{2.62}$$

since $u > 0$ and $w_0^T \geq 0, w_0 \neq 0$ we have that $w_0^T u$ is a real positive value. Furthermore c is a positive constant less that 1. The inequality is then

$$\rho\left(\frac{\partial I}{\partial p}(p)\right) < 1 \tag{2.63}$$

this concludes the proof. ■

From this last proposition we can conclude that:

i. if f is a c-contractive function then there exists a and b such that $f(a) > a$, $f(b) < b$ and $b > a$. This guarantees the existence of a positive fixed point from Theorem 2.1.1.

ii. if f is a strictly concave function such that con $f(0) \geq 0$, $\rho\left(\frac{\partial f}{\partial x}(0)\right) > 1$ and $\rho\left(\frac{\partial f}{\partial x}(\bar{x})\right) < 1$ for some $\bar{x} \in int(\mathbb{R}_+^n)$ then from Theorem 2.2.1 there exists a positive fixed point in $int(\mathbb{R}_+^n)$. However f is not c-contractive since in a neighbourhood of 0, $\rho\left(\frac{\partial f}{\partial x}(x)\right) \geq 1$. For example the function $f(x) = \sqrt{x} + 5 \in \mathbb{R}_+$ is strictly concave and from Theorem 2.2.1 admits a unique positive fixed point but f is not c-contractive since $f'(0^+) \to +\infty$ and this violates the condition of Proposition 2.3.2.

Although condition ii seems to state that c-contractive functions are less general than strictly concave function with $\rho\left(\frac{\partial f}{\partial x}(0)\right) > 1$, the next example shows that c-contractive does not imply concavity, i.e., they are different classes of functions.

Example 2.3.1. *The following example ([12]) represents a non-concave c-contractive function.*

$$I(p) = \begin{cases} p^2 + \dfrac{1}{100}, & 0 \leq p \leq \dfrac{1}{4} \\ 0.5p - \dfrac{1}{16} + \dfrac{1}{100}, & p > \dfrac{1}{4} \end{cases} \tag{2.64}$$

Proposition 2.3.3. *Let $I : \mathbb{R}_+^n \to \mathbb{R}_+^n$ be a differentiable increasing function such that $\forall p \in \mathbb{R}_+^n$. If the following holds $\forall p \in \mathbb{R}_+^n$*

1. $I(p) > 0$
2. $\rho\left(\frac{\partial I}{\partial p}(p)\right) \leq \zeta_0 < 1$
3. $\frac{\partial I}{\partial p}(p)$ *is irreducible*

Then I is c-contractive with $c = \zeta_0 + \frac{1-\zeta_0}{2}$.

Proof. From Taylor's approximation

$$I(p+\varepsilon v) = I(p) + \frac{\partial I}{\partial p}(p)\varepsilon v + h(\varepsilon v).$$

Now let $v = \gamma v_0$ with $\gamma > 0$ and v_0 is the right Perron-Frobenius eigenvector of $\frac{\partial I}{\partial p}(p)$ corresponding to the eigenvalue $\rho\left(\frac{\partial I}{\partial p}(p)\right)$. The last equation becomes

$$I(p+\varepsilon v) = I(p) + \varepsilon\rho\left(\frac{\partial I}{\partial p}(p)\right)\gamma v_0 + h\left(\varepsilon\gamma v_0\right). \tag{2.65}$$

Now I is c-contractive if for some $c \in [0,1)$ we have

$$I(p+\varepsilon v) \leq I(p) + c\varepsilon v \tag{2.66}$$

Now, from (2.65) we can guarantee the Contractive condition if

$$\varepsilon\rho\left(\frac{\partial I}{\partial p}(p)\right)\gamma v_0 + h\left(\varepsilon\gamma v_0\right) \leq \left(\zeta_0 + \frac{1-\zeta_0}{2}\right)\varepsilon\gamma v_0, \tag{2.67}$$

we define c as

$$c := \zeta_0 + \frac{1-\zeta_0}{2}. \tag{2.68}$$

Clearly $1 > c > \zeta_0$. Inequality (2.67) is then

$$\gamma\left(\left(c - \rho\left(\frac{\partial I}{\partial p}(p)\right)\right)\varepsilon v_0 + \frac{h\left(\varepsilon\gamma v_0\right)}{\gamma}\right) \geq 0. \tag{2.69}$$

By assumptions on $\rho\left(\frac{\partial I}{\partial p}(p)\right)$ we have that $c - \rho\left(\frac{\partial I}{\partial p}(p)\right) > 0$ is a positive scalar for all $p \in \mathbb{R}^n_+$. From Taylor's approximation $h \to 0$ if $\gamma \to 0$, inequality (2.69) can be satisfied only if $v_0 > 0$, that is $\frac{\partial I}{\partial p}(p)$ is irreducible.
In conclusion, under conditions $\rho\left(\frac{\partial I}{\partial p}(p)\right) \leq \zeta_0 < 1$ and $\frac{\partial I}{\partial p}(p)$ irreducible for all $p \in \mathbb{R}^n_+$, I is contractive, i.e.

$$I(p+\varepsilon v) \leq I(p) + c\varepsilon v \tag{2.70}$$

with $v = \gamma v_0$, for an appropriate $\gamma > 0$ and c defined as is (2.68). ■

2.4 Extension of results to subhomogeneous vector fields

The following theorem widen the class of c-contractive functions to the case where there might exists some point such that $\rho\left(\frac{\partial f}{\partial x}(x)\right) > 1$.

Proposition 2.4.1. *Consider system* (2.2). *Let $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a differentiable increasing vector field and such that $f(0) \geq 0$. Let $\frac{\partial f}{\partial x}(x) \geq 0\ \forall x \in \mathbb{R}^n_+$ be irreducible for all $x \in \mathbb{R}^n_+$. Then*

1. *if $f(0) > 0$ or $\rho\left(\frac{\partial f}{\partial x}(0)\right) = \lambda_0 > 1 \Rightarrow\ \exists\, a \in \mathbb{R}^n_+$, $a > 0$ such that $f(a) > a$*

2. *if $\exists\, x_0 \in \mathbb{R}^n_+$ such that $\rho\left(\frac{\partial f}{\partial x}(x_0)\right) = \zeta_0 < 1$ and $\rho\left(\frac{\partial f}{\partial x}(x)\right) \leq \zeta_0\ \forall x > x_0 \Rightarrow\ \exists\, b \in \mathbb{R}^n_+$, $b > a$ such that $f(b) < b$, and $a > 0$ is such that $f(a) > a$.*

Proof of 1. If $f(0) = f_0 > 0$ from the assumption for f to be increasing, then $f(x) > f_0$ for all $x > 0$. For example if $a = \frac{f_0}{2} > 0$ clearly $f(a) > f_0 > a$.
The existence of a such that $f(a) > a$ in the case $\rho\left(\frac{\partial f}{\partial x}(0)\right) = \lambda_0 > 1$ has already been shown in the proof of part 1 of Theorem 2.2.1, the proof is based on Taylor's approximation and the hypothesis of concavity is not used. ■

Proof of 2. Now suppose there exists $\bar{x} \in \mathbb{R}^n_+$ such that $\rho\left(\frac{\partial f}{\partial x}(\bar{x})\right) = \zeta_0 < 1$ and there exists $a \in \mathbb{R}^n_+$, $a > 0$ such that $f(a) > a$. Let $g : \mathbb{R}^n_+ \to \mathbb{R}^n$ be defined as $g(x) = f(x) - x$.
From Taylor's theorem and from differentiability assumptions for f, g is differentiable and for every $i = 1, \ldots, n$, the following holds

$$g_i(x) = g_i(x_0) + \nabla g_i(x_0)(x - x_0) + h_i(x - x_0), \tag{2.71}$$

where $h_i : \mathbb{R}^n \to \mathbb{R}$ is such that

$$\lim_{x \to x_0} \frac{h_i(x - x_0)}{\|x - x_0\|} = 0 \tag{2.72}$$

which by joining all equations yields

$$g(x) = g(x_0) + \frac{\partial g}{\partial x}(x_0)(x - x_0) + h(x - x_0). \tag{2.73}$$

This second part of the theorem states that there exists $b > a$ such that $f(b) < b$, that is $g(b) < 0$.
Let us define $x_1 \in \mathbb{R}^n_+$, $x_1 > x_0$ as

$$x_1 = x_0 + \gamma_0 v_0, \quad \gamma_0 > 0$$

where $v_0 > 0$ is the positive Perron-Frobenius eigenvalue of the Jacobian matrix of f corresponding to the eigenvalue ζ_0, i.e., $\frac{\partial f}{\partial x}(x_0)v_0 = \rho\left(\frac{\partial f}{\partial x}(x_0)\right)v_0$. Equivalently, that is

$$\frac{\partial g}{\partial x}(x_0)v_0 = (-1+\zeta_0)v_0 = -\varepsilon_0 v_0$$

where $\varepsilon_0 > 0$ since $\zeta_0 < 1$. Now, from Taylor's approximation and with the given choice of x_1, the following holds

$$g(x_1) = g(x_0) + \frac{\partial g}{\partial x}(x_0)(x_1 - x_0) + h_0(x_1 - x_0),$$

which yields

$$g(x_1) - g(x_0) = -\varepsilon_0\gamma_0 v_0 + h_0(x_1 - x_0) \tag{2.74}$$

and since $\frac{1}{\gamma_0}h_0(x_1 - x_0) \to 0$ for $\gamma_0 \to 0$ we can make $-\varepsilon_0\gamma_0 v_0 + h(x_1 - x_0) < 0$ for an appropriate small $\gamma_0 \equiv \hat{\gamma}_0 > 0$. This means that $g(x_1) < g(x_0)$.
Furthermore, if $g(x_0) - \varepsilon_0\gamma_0 v_0 + h_0(x_1 - x_0) < 0$ we are done since the proof will be concluded choosing $b \equiv x_1$ otherwise we iterate the procedure defining $x_2 > x_1$ as

$$x_2 = x_1 + \gamma_1 v_1, \quad \gamma_1 > 0$$

where $v_1 > 0$ is the positive Perron-Frobenius eigenvalue of the Jacobian matrix of f corresponding to the eigenvalue $\rho\left(\frac{\partial f}{\partial x}(x_1)\right) \le \zeta_0$ by assumption. This yields a similar expression to (2.74)

$$g(x_2) - g(x_1) = -\varepsilon_1\hat{\gamma}_1 v_1 + h_1(x_2 - x_1)$$

and in general,

$$g(x_{i+1}) - g(x_i) = -\varepsilon_i\hat{\gamma}_i v_i + h_i(x_{i+1} - x_i) \tag{2.75}$$

it is easy to show that $\varepsilon_0 \le \varepsilon_i$. Now, let us assume we have iterated the equations till the N^{th} order. The following holds

$$g(x_N) - g(x_0) = \sum_{i=0}^{N-1} -\varepsilon_i\hat{\gamma}_i v_i + h_i(x_{i+1} - x_i) < 0. \tag{2.76}$$

We want to show that x_N do not converge for $N \to +\infty$. In fact from

$$\begin{cases} x_1 = x_0 + \hat{\gamma}_0 v_0 \\ \vdots \\ x_N = x_{N-1} + \hat{\gamma}_{N-1} v_{N-1} \end{cases} \tag{2.77}$$

if $x_N \to \bar{x}$ for $N \to +\infty$ we can always choose X_M as

$$x_M = \bar{x} + \hat{\gamma}_N v_N > \bar{x}$$

since $g(x_N) \simeq g(\bar{x})$, the new sequence converges to $x_M > \bar{x}$. This means that it is possible to choose a sequence of x_i such that it does not converge for $N \to \infty$, thus we can assume $x_i \to +\infty$ if $N \to +\infty$ $\forall i = 1, \dots, n$. Therefore, from

$$x_N - x_0 = \sum_{i=0}^{N-1} \hat{\gamma}_i v_i$$

for N big enough each component of the vector tends to a big number, i.e. from

$$g(x_N) - g(x_0) = \sum_{i=0}^{N-1} -\varepsilon_i \hat{\gamma}_i v_i + h_i(x_{i+1} - x_i) < 0 \tag{2.78}$$

and for big N, the sum $\sum_{i=0}^{N-1} -\varepsilon_i \hat{\gamma}_i v_i$ if less than $-g(x_0) - \sum_{i=0}^{N-1} h_i(x_{i+1} - x_i)$, hence $g(x_N) < 0$. This concludes the proof since b can be chosen equal to X_N. ■

Remark. *Under the conditions of this proposition the existence of a positive fixed point is guaranteed (see Theorem 2.1.1). However the fixed point might not be unique since for all* $x < x_0$ *the function can grow its spectral radius pass one and then again lower it.*

Under the assumptions of Proposition 2.4.1, the next theorem adds a sufficient condition to guarantee the uniqueness of the positive fixed point. As already shown in Proposition 1.2.3 the class of subhomogeneous vector field includes (but it is strictly larger than) the class of concave vector fields (Krause [20]).

Theorem 2.4.1. *Consider system* (2.2). *Let* $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *be a differentiable increasing vector field and such that* $f(0) \geq 0$. *Let* $\frac{\partial f}{\partial x}(x) \geq 0\ \forall x \in \mathbb{R}^n_+$ *be irreducible for all* $x \in \mathbb{R}^n_+$. *Then*

1. *if* $f(0) > 0$ *or* $\rho\left(\frac{\partial f}{\partial x}(0)\right) = \lambda_0 > 1 \Rightarrow\ \exists\, a \in \mathbb{R}^n_+$, $a > 0$ *such that* $f(a) > a$
2. *if* $\exists\, x_0 \in \mathbb{R}^n_+$ *such that* $\rho\left(\frac{\partial f}{\partial x}(x_0)\right) = \zeta_0 < 1$ *and* $\rho\left(\frac{\partial f}{\partial x}(x)\right) \leq \zeta_0\ \forall x > x_0 \Rightarrow\ \exists\, b \in \mathbb{R}^n_+$, $b > a$ *such that* $f(b) < b$, *and* $a > 0$ *is such that* $f(a) > a$

Then there exists a positive fixed point. If, in addition, f *is strictly subhomogeneous of degree* $0 < \tau \leq 1$ *then the fixed point is unique.*

Proof. The existence has already been shown. Let us show the uniqueness part, which is essentially the same as in 2.1.2 with just a little variation.
From the definition of subhomogeneous (1.9), f is strictly subhomogeneous of degree $\tau > 0$ if

$$f(\alpha x) > \alpha^{\tau} f(x) \tag{2.79}$$

for all $x \in int(\mathbb{R}^n_+)$ and for all $\alpha \in (0,1)$. Let $g : \mathbb{R}^n_+ \to \mathbb{R}^n : x \mapsto -x + f(x)$.
If f is strictly subhomogeneous of degree $\tau \leq 1$ so is g. To show this note that the function

$$h(x) = -x, \quad x \geq 0 \tag{2.80}$$

is subhomogeneous of degree $\tau \leq 1$. In fact from definition, h is subhomogeneous of degree $\tau > 0$ if

$$h(\alpha x) \geq \alpha^{\tau} h(x), \quad \forall \alpha \in [0,1] \tag{2.81}$$

where $h(\alpha x) = -\alpha x$ and $\alpha^{\tau} h(x) = -\alpha^{\tau} x$. Which imposing the inequality in (2.81) yields

$$-\alpha x \geq -\alpha^{\tau} x \Rightarrow \alpha x \leq \alpha^{\tau} x$$

since $\alpha \in [0,1]$ and $x \geq 0$ this is satisfied for all $\tau \in (0,1]$.
From the following relationship we can see that g is subhomogeneous of degree $\tau \leq 1$.

$$g(\alpha x) = -\alpha x + f(\alpha x) \overbrace{=}^{(2.80)} h(\alpha x) + f(\alpha x) \overbrace{\geq}^{(2.81)} \alpha^{\tau} h(x) + f(x) \overbrace{>}^{(2.79)} \alpha^{\tau} h(x) + \alpha^{\tau} f(x) = \alpha^{\tau} g(x)$$

therefore $g(\alpha x) > \alpha^{\tau} g(x)$ for all $\alpha \in (0,1)$ given $\tau \leq 1$ is the degree of f.
Now, to show uniqueness of the fixed point, suppose $x > 0$ is any fixed point of f. Suppose $y > 0$ and $g(y) \geq 0$. Let

$$\alpha = min\left\{\frac{x_j}{y_j}, j = 1, \ldots, n\right\} = \frac{x_r}{y_r}.$$

Then $\alpha > 0$ because $x > 0$ and $y > 0$. If $a \geq 1$ the $y \leq x$. Otherwise let $w = \alpha y$. Since g is strictly subhomogeneous and $g(y) \geq 0$ we have that $g(\alpha y) > \alpha^{\tau} g(y)$ for $0 < \alpha < 1$ this imply $g(w) > 0$. Then $w \leq x$ and $w_r = x_r$, so $g_r(x) - g_r(w) = f_r(x) - f_r(w) \geq 0$ because f is increasing. But this implies $0 = g_r(x) \geq g_r(w) > 0$, a contradiction. Thus $y > 0$ and $g(y) \geq 0$ implies $y \leq x$.
Now if $y > 0$ is a fixed point of f then, since $g(x) = 0$, the same argument with the roles of x and y reversed gives $x \leq y$, so $y = x$. ■

We would like to weaken the monotonicity property for f while still guaranteeing the existence of a unique positive fixed point. Recalling definition (1.2) of spectral abscissa[1] we now give the following:

Theorem 2.4.2. *Consider system* (2.2)*. Let* $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *be a differentiable, strictly sub-homogeneous vector field of degree* $0 < \tau \leq 1$ *and such that* $f(0) \geq 0$*. Let* $\frac{\partial f}{\partial x}(x)$ *be Metzler and irreducible for all* $x \in \mathbb{R}^n_+$*. If the following conditions hold*

1. *if* $f(0) > 0$ *or* $\mu\left(\frac{\partial f}{\partial x}(0)\right) = \lambda_0 > 1 \Rightarrow\ \exists\, a \in \mathbb{R}^n_+,\ a > 0$ *such that* $f(a) > a$
2. *if* $\exists\, x_0 \in \mathbb{R}^n_+$ *such that* $\mu\left(\frac{\partial f}{\partial x}(x_0)\right) = \zeta_0 < 1$ *and* $\mu\left(\frac{\partial f}{\partial x}(x)\right) \leq \zeta_0\ \forall x > x_0 \Rightarrow\ \exists\, b \in \mathbb{R}^n_+$*,* $b > a$ *such that* $f(b) < b$*, and* $a > 0$ *is such that* $f(a) > a$
3. *if* f *is increasing in* $\Omega = \{x \in \mathbb{R}^n_+ : 0 \leq x \leq \hat{b}\}$*, and* $\hat{b}$ *is a point such that* $f(\hat{b}) < \hat{b}$

Then there exists a unique positive fixed point in $int(\mathbb{R}^n_+)$*.*

Proof. We will show the uniqueness, adapted from (Bokharaie et al. [5]), since existence is implied by the previous Theorem 2.4.1. System (2.2) can be written as

$$\dot{x}(t) = g(x) \tag{2.82}$$

where g is defined as usual and Δ has been ignored. To show the uniqueness of the fixed point, an equilibria for $\dot{x}(t) = g(x)$, we need the assumption of Jacobian matrix to be Metzler and irreducible for all $x \in \mathbb{R}^n_+$.
By contradiction let us suppose that there are two distinct equilibria $p \in \Omega,\ q \in int(\mathbb{R}^n_+)\backslash\Omega$. The Jacobian matrix evaluated at each point $g(b) \leq 0,\ b > 0$ is Hurwitz, in fact strict sub-homogeneity of g yields (Lemma 1.2.1)

$$\frac{\partial g}{\partial x}(x)x < \tau g(x), \quad \forall x > 0 \tag{2.83}$$

evaluated in b yields

$$\left.\frac{\partial g}{\partial x}(x)x\right|_{x=b} < \tau g(b) \leq 0 \tag{2.84}$$

multiplying both sides by w_b^T, which is the positive eigenvector corresponding to the eigenvalue with the maximum real part $\mu\left(\frac{\partial g}{\partial x}(b)\right)$, we obtain

$$\mu\left(\frac{\partial g}{\partial x}(b)\right) w_b^T b < 0 \tag{2.85}$$

[1] For the sake of simplicity, $\mu(A) = max\{Re(\lambda), \lambda \in \Lambda(A)\}$

thus $\mu\left(\frac{\partial g}{\partial x}(b)\right) < 0$ since $w_b^T b > 0$. In particular, this means that the Jacobian matrix evaluated in p and q is Hurwitz. From the irreducibility of g and since it is Metzler, there exists two positive eigenvectors $v_p, v_q > 0$ corresponding to the maximal real part eigenvalues $\mu\left(\frac{\partial g}{\partial x}(p)\right) < 0$ and $\mu\left(\frac{\partial g}{\partial x}(q)\right) < 0$, that is,

$$\frac{\partial g}{\partial x}(p)v_p = \mu\left(\frac{\partial g}{\partial x}(p)\right)v_p < 0,$$

$$\frac{\partial g}{\partial x}(q)v_q = \mu\left(\frac{\partial g}{\partial x}(q)\right)v_q < 0.$$

We can assume that

$$\max_i \frac{q_i}{p_i} > 1 \quad \forall i = 1,\ldots,n$$

Since g is differentiable, it follows from Taylor's theorem that

$$\begin{cases} g(p+\gamma v_p) = g(p) + \gamma\frac{\partial g}{\partial x}(p)v_p + h_p(\gamma v_p) \\ g(q-\gamma v_q) = g(q) - \gamma\frac{\partial g}{\partial x}(q)v_q + h_q(\gamma v_q) \end{cases} \tag{2.86}$$

where $h_{p,q} \to 0$ as $\gamma \to 0^+$ and $g(p) = g(q) = 0$. Therefore, there exists an appropriate small $\hat{\gamma}$ such that $g(p+\hat{\gamma}v_p) > 0$ and $g(q-\hat{\gamma}v_q) < 0$. Define $v = p+\hat{\gamma}v_p$, $w = q-\hat{\gamma}v_q$. Defining ξ as

$$\xi := \max_i \frac{w_i}{v_i} = \frac{w_r}{v_r}.$$

Note that with an appropriate $\hat{\gamma} > 0$ we can ensure $\xi > 1$. From this choices the following holds:

i. $\xi v \geq w$ and $\xi v_r = w_r$

ii. $g(\xi v) \leq \xi^\tau g(v)$

As g is cooperative it satisfies Kamke condition[2] (see [16]), it follows from (i) that $g_r(\xi v) \geq g_r(w) > 0$. On the other hand, it follows from (ii) that $g_r(\xi v) \leq g_r(w) < 0$.
This is a contradiction, which shows that there can only be one equilibrium in $int(\mathbb{R}^n_+)$. ■

2.4.1 Proof of convergence

Theorem 2.4.3. *Consider system* (2.2). *Under assumptions of Theorem 2.4.1 or Theorem 2.4.2 there exists a unique* $x^* \in int(\mathbb{R}^n_+)$ *such that* $x(t,x_0) \to x^*$ *for all* $x_0 \in \mathbb{R}^n_+\backslash\{0\}$.

[2] if for each i, $f_i(a) \leq f_i(b)$ for any two points a and b in and open subset satisfying $a \leq b$ and $a_i = b_i$

Proof. We will give proof for Theorem 2.4.2 only, Theorem 2.4.1 is a sub case.
From strict subhomogeneity of f of degree $0 < \tau \leq 1$, without loss of generality we can assume $\tau = 1$

$$\begin{aligned} f(\lambda x) &< \lambda^\tau f(x) \leq \lambda f(x), \quad \forall \lambda > 1 \\ f(\alpha x) &> \alpha^\tau f(x) \geq \alpha f(x), \quad 0 < \alpha < 1 \end{aligned} \tag{2.87}$$

subtracting $-\lambda x$ from both sides of the first of (2.87) and $-\alpha x$ from both sides of the second we obtain

$$\begin{aligned} -\lambda x + f(\lambda x) &< -\lambda x + \lambda f(x), \quad \forall \lambda > 1 \\ -\alpha x + f(\alpha x) &> -\alpha x + \alpha f(x), \quad 0 < \alpha < 1 \end{aligned}$$

which are equivalent to

$$\begin{aligned} g(\lambda x) &< \lambda g(x), \quad \forall \lambda > 1, \\ g(\alpha x) &> \alpha g(x), \quad 0 < \alpha < 1. \end{aligned}$$

From $g(x^*) = 0$ and $\mu = \left(\frac{\partial g}{\partial x}(x^*)\right) < 0$, i.e. $\frac{\partial g}{\partial x}(x^*)$ is Hurwitz in x^* (see proof of Theorem 2.4.2), we can find $b > x^*$ such that $g(b) < 0$ and $a < x^*$ such that $g(a) > 0$. From Taylor's approximation it can be for example $b = x^* + \gamma v_0$, $a = x^* - \gamma v_0$ for an appropriate small γ. From strict subhomogeneity and since $g(b) < 0$ and $g(a) > 0$ the following two holds

$$\begin{aligned} g(\lambda b) &< \lambda g(b) < 0, \quad \forall \lambda > 1 \\ g(\alpha a) &> \alpha g(a) > 0, \quad \forall \alpha \in (0,1). \end{aligned} \tag{2.88}$$

Now, if $\frac{\partial g}{\partial x}(x)$ is cooperative and irreducible $\forall x > 0$, from Smith [24] we can conclude that if $g(x) < 0$ then $x(t,x)$ is strictly decreasing and tends to x^*, analogously if $g(x) > 0$ $x(t,x)$ is strictly increasing and tends to x^* since x^* is the unique equilibrium point, formally

$$\begin{aligned} g(x) < 0 &\Rightarrow x(t,x) \to x^* \text{ and } \dot{x}(t,x) < 0 \quad \forall t > 0, \\ g(x) > 0 &\Rightarrow x(t,x) \to x^* \text{ and } \dot{x}(t,x) > 0 \quad \forall t > 0. \end{aligned}$$

Now $\forall x_0 \in int(\mathbb{R}^n_+)$ and $x_0 \neq x^*$, we have two possible cases (which can hold simultaneously):

1. if there exists i such that $x_{0i} \geq x^*_i$ then there exist $\bar{\lambda} > 1$ such that $\bar{\lambda} b > x_0$, and necessarily $g(\bar{\lambda} b) < 0$ as shown in the first of (2.88).

2. if there exists j such that $x_{0j} < x^*_j$ then there exist $\bar{\alpha} < 1$ such that $\bar{\alpha} a < x_0$, and necessarily $g(\bar{\alpha} a) > 0$ as shown in the second of (2.88).

That is $\forall x_0 \in int(\mathbb{R}^n_+)$, there exists $\bar{\alpha} < 1$ and $\bar{\gamma} >$ such that

$$\bar{\alpha} a \leq x_0 \leq \bar{\gamma} b \tag{2.89}$$

which implies, from cooperativity and irreducibility

$$x(t, \bar{\alpha} b) < x(t, x_0) < x(t, \bar{\gamma} b) \tag{2.90}$$

thus $x(t, x_0) \to x^*$ since $x(t, \bar{\alpha} b) \to x^*$ and $x(t, \bar{\gamma} b) \to x^*$.
The same proof can be done using a Lyapunov's functions. Let us choose the following Lyapunov's function

$$V(x) = \frac{1}{2}\|x - x^*\|^2. \tag{2.91}$$

Clearly $V(x) = 0$ if and only if $x = x^*$. Let $x(t, x_0)$ be the evolution of the solution with initial condition x_0 and $t \geq 0$.

$$\dot{V}\left(x(t, \bar{\lambda} b)\right) = \left(x(t, \bar{\lambda} b) - x^*\right)^T \dot{x}(t, \bar{\lambda} b). \tag{2.92}$$

The sets $\Omega_+ := \{x \in int(\mathbb{R}^n_+) : x \geq x^*\}$ and $\Omega_- := \{x \in int(\mathbb{R}^n_+) : x \leq x^*\}$ are two invariant sets. In fact under our assumptions on f, system (2.2) is cooperative, therefore monotone (see [4, 24]), that is

$$x_0 \leq y_0 \Rightarrow x(t, x_0) \leq x(t, y_0),$$

that is for Ω_-,

$$x_0 \leq x^* \Rightarrow x(t, x_0) \leq x(t, x^*) = x^*$$

and for Ω_+,

$$x^* \leq x_0 \Rightarrow x^* = x(t, x^*) \leq x(t, x_0).$$

For our purposes this means $x(t, \bar{\lambda} b) - x^* \in \Omega_+ \ \forall t \geq 0$, therefore $x(t, \bar{\lambda} b) - x^* \geq 0 \ \forall t \geq 0$. Since $g(\bar{\lambda} b) < 0$ it follows from (2.4.1) that $\dot{x}(t, \bar{\lambda} b) < 0$ and from $x^* < \bar{\lambda} b \Rightarrow x^* < x(t, \bar{\lambda} b)$ we have

$$\dot{V}(x(t, \bar{\lambda} b)) < 0, \quad \forall t > 0. \tag{2.93}$$

This means that $V(x(t, \bar{\lambda} b)) \to 0$, in other words $x(t, \bar{\lambda} b) \to x^*$ if $t \to \infty$.
Analogously,

$$\dot{V}\left(x(t, \bar{\alpha} a)\right) = \left(x(t, \bar{\alpha} a) - x^*\right)^T \dot{x}(t, \bar{\alpha} a). \tag{2.94}$$

Since $g(\bar{\alpha} a) > 0$ it follows from (2.4.1) that $\dot{x}(t, \bar{\alpha} b) > 0$ and from $\bar{\alpha} a < x^* \Rightarrow x(t, \bar{\alpha} b) < x^*$

we have

$$\dot{V}(x(t,\bar{\alpha}b)) < 0, \ \ \forall t > 0. \tag{2.95}$$

Therefore $x(t,\bar{\alpha}b) \to x^*$.
This concludes the proof since (2.93) and (2.94) implies that $\forall x_0 \in int(\mathbb{R}^n_+)$, $x(t,x_0) \to x^*$. ■

We will now give a sample function that satisfies all assumptions of the previous theorems.

Example 2.4.1. *Consider the function in one variable*

$$f(x) = \alpha x e^{-\frac{x}{\beta}} \tag{2.96}$$

this function is strictly subhomogeneous of degree $\tau = 1$*. In fact*

$$f(\eta x) = \alpha \eta x e^{-\frac{\eta x}{\beta}}$$

for $0 < \eta < 1$ *and* $x > 0$ *the term* $e^{-\frac{\eta x}{\beta}}$ *is decreasing, thus*

$$f(\eta x) = \alpha \eta x e^{-\frac{\eta x}{\beta}} > \alpha \eta x e^{-\frac{x}{\beta}} = \eta f(x)$$

in other words, f *is strictly subhomogeneous of degree* $\tau = 1$*.*
The next condition to be satisfied is on its spectral abscissa. Some calculation yields

$$\alpha = \mu\left(\frac{\partial f}{\partial x}(0)\right)$$

and

$$0 = \mu\left(\frac{\partial f}{\partial x}(\beta)\right)$$

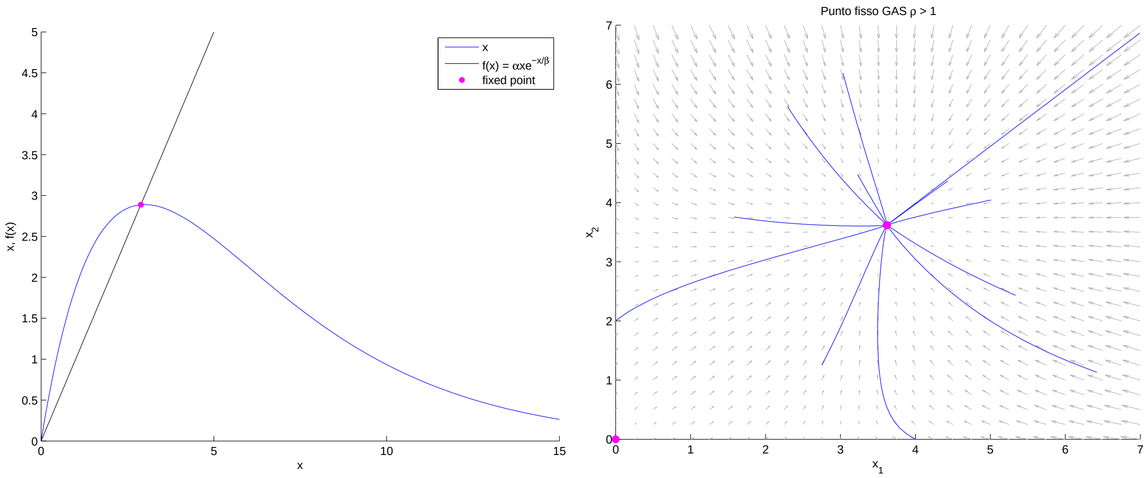
choosing $\alpha > 1$ *and since* f *is decreasing* $\forall x > \beta$*, i.e.* $f'(x) < 0$ *conditions on the spectral abscissa are satisfied.*
Imposing

$$\frac{\partial f}{\partial x}(x) = \alpha e^{-\frac{x}{\beta}} - \frac{\alpha}{\beta} x e^{-\frac{x}{\beta}} \geq 0$$

we obtain $x \leq \beta$*. Now, the fixed point* $f(x^*) = x^*$ *is*

$$x^* = \beta \log \alpha$$

thus if $x^* < \beta$ *there exists* $b > 0$ *such that* f *is increasing in* $\Omega = \{x \in \mathbb{R}_+ : 0 \leq x \leq b\}$*. This* *happens for $\alpha < e$. Choosing $\alpha = e - 0.1$ and $\beta = 3$ we obtain the the function depicted in figure 2.1a, which is clearly neither concave nor monotone. The evolution of the solution converges to x^* whenever $x_0 > 0$.*



(a) Subhomogeneous vector field $f(x) = \alpha x e^{-\frac{x}{\beta}}$ (b) Trajectory evolution of system (2.97)

Figure 2.1 Subhomogeneous example

Now let us consider a bi-dimensional case:

$$\begin{cases} \dot{x}_1 = -x_1 + \alpha x_1 e^{-\frac{x_1}{\beta}} + \dfrac{x_2}{1+x_2} \\ \dot{x}_2 = -x_2 + \alpha x_2 e^{-\frac{x_2}{\beta}} + \dfrac{x_1}{1+x_1} \end{cases} \tag{2.97}$$

for $x \geq 0$ the vector field f

$$f(x) = \begin{pmatrix} \alpha x_1 e^{-\frac{x_1}{\beta}} + \dfrac{x_2}{1+x_2} \\ \alpha x_2 e^{-\frac{x_2}{\beta}} + \dfrac{x_1}{1+x_1} \end{pmatrix} \tag{2.98}$$

is strictly subhomogeneous of degree $\tau = 1$ since $\frac{x_i}{1+x_i}$ is strictly subhomogeneous. Its Jacobian matrix

$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} \alpha e^{-\frac{x_1}{\beta}} - \frac{\alpha}{\beta} x_1 e^{-\frac{x_1}{\beta}} & \frac{1}{(1+x_2)^2} \\ \frac{1}{(1+x_1)^2} & \alpha e^{-\frac{x_2}{\beta}} - \frac{\alpha}{\beta} x_2 e^{-\frac{x_2}{\beta}} \end{bmatrix} \tag{2.99}$$

which is clearly Metzler and irreducible for all $x \geq 0$ and $\mu\left(\frac{\partial f}{\partial x}(x)\right) \to 0$ for $x_{1,2} \to +\infty$. The trajectory of the solutions of system (2.97) to fixed point in the interior of $\mathbb{R}^n_+$ whenever

$x \neq 0$. *This is shown in figure 2.1b.*

Chapter 3

Applications to Distributed systems

3.1 Distributed system dynamics

We consider a directed graph, the influence graph, associated to the vector field of a system where the nodes represents the states' variables. We only consider pairwise interactions in which the state of a node propagates to its neighbours following the direction of the edges. The incoming influences at a node obey the principle of linear superposition of the effects. In addition, the network includes first order degradation terms on the diagonal, depending on the degradation rate constants, denoted δ_i, $i = 1, \dots, n$ (see Altafini [1]). Letting $x_i \in \mathbb{R}^n_+$ be the state of the i^{th} node, the system can be written as

$$\dot{x}_i = -\delta_i x_i + f_i(x) = -\delta_i x_i + \sum_{j \in \mathcal{N}_i} f_{ij}(x_j), \;\; i = 1, \dots, n \tag{3.1}$$

where $\mathcal{N}_i$ is the set of incoming neighbours of node i, and $f_{ij}(x_j)$ is the influence exerted by the j^{th} node on the i^{th} node. We assume that a node j exerts the same form of influence on all its neighbours, up to a scaling constant which corresponds to the weight of the edge connecting j with i. If $A = [a_{ij}] \geq 0$ is the weighted adjacency matrix of the network, and $\varphi_j(x_j)$ is the functional form of the interaction from node j to all its neighbours, then we can write

$$f_{ij}(x_j) = a_{ij} \varphi_j(x_j) \tag{3.2}$$

By defining $f_i(x)$ as $f_i(x) = \sum\limits_{j \in \mathcal{N}_i} f_{ij}(x_j)$ the system has the same structure considered in the previous section. To apply the main theorems of that sections we need $f = (f_1, \dots, f_n)^T$ to be monotone, strictly concave and with irreducible Jacobian. This implies $\varphi_i(x_i)$ monotone strictly concave and $\varphi_i(0) = 0 \; \forall i$.

Explicating the weight of the interaction also in the adjacency matrix A, the dynamics of

(3.1) can be represented in matrix form as

$$\dot{x} = -\Delta x + A\varphi(x) \tag{3.3}$$

where $\varphi(x) = [\varphi_1(x_1), \ldots, \varphi_n(x_n)]^T$. Since

$$\frac{\partial}{\partial x} A\varphi(x) = A \begin{bmatrix} \frac{\partial \varphi_1(x_1)}{\partial x_1} & 0 & \ldots & 0 \\ 0 & \frac{\partial \varphi_2(x_2)}{\partial x_2} & \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ldots & \frac{\partial \varphi_n(x_n)}{\partial x_n} \end{bmatrix}. \tag{3.4}$$

Then if A is irreducible we can apply Theorem 2.2.3 of previous section. For distributed systems we would like to investigate what happens if A is reducible.

3.1.1 Reducible Adjacency matrix

If A is reducible, from Proposition 1.2.5 there exists a permutation matrix that transforms A into its normal form (1.16)

$$\bar{A} = \Pi^T A \Pi = \left[\begin{array}{cccc|cccc} \bar{A}_{1,1} & & & & & & & \\ 0 & \bar{A}_{2,2} & & & & 0 & & \\ \vdots & \ldots & \ddots & & & & & \\ 0 & 0 & \ldots & \bar{A}_{h,h} & & & & \\ \hline \star & \star & \ldots & \star & \bar{A}_{h+1,h+1} & & & \\ \star & \star & \ldots & \star & \star & \bar{A}_{h+2,h+2} & & \\ \star & \star & \ldots & \vdots & \vdots & \ldots & \ddots & \\ \star & \star & \ldots & \star & \star & \star & \ldots & \bar{A}_{kk} \end{array}\right]. \tag{3.5}$$

The next theorem shows that to guarantee the existence and convergence to a positive fixed point it is necessary to guarantee existence and convergence to a positive fixed point for all isolated systems while the other systems should not be unstable.

Theorem 3.1.1. *Consider the distributed system (3.3). Let us assume $A \in \mathbb{R}_+^{n\times n}$ to be reducible. Let Σ_i be the subsystems for $i = 1, \ldots, k$. Let $\mathcal{N}_i$ contain the indexes of i^{th} subsystem, i.e $j \in \mathcal{N}_i$ if $x_j \in \mathbb{R}_+^n$ is part of subsystem Σ_i. We also assume that the isolated subsystems $\{\Sigma_1, \ldots, \Sigma_h\}$ satisfies the conditions of Theorem 2.4.2 for existence of a positive fixed point. Under these assumptions, let $x^*_{\Sigma 1}, \ldots x^*_{\Sigma h}$ be the positive fixed point for each isolated subsystem. Let $x_0 = (x^T_{\Sigma 1,0}, \ldots, x^T_{\Sigma h,0}, x^T_{\Sigma h+1,0}, \ldots x^T_{\Sigma k,0})^T \in \mathbb{R}^n_+$ be the initial condition. The whole* *system (3.3) converges to a positive fixed point* $\bar{\mathbf{x}} \in int(\mathbb{R}^n_+)$ *if and only if*

1. $x_{\Sigma 1}(t, x_{\Sigma i,0}) \to x^*_{\Sigma i}$ *for* $i = 1, \dots, h$
2. *subsystems* $h+1, \dots, h+n$ *are asymptotically stable.*

Before seeing the proof we would like to make a little observation.

Remark. *Theorem 3.1.1 means that if all subsystems are stable we can guarantee convergence to the interior of* $\mathbb{R}^n_+$ *for the whole network if we manage to guarantee convergence of the isolated subsystems to a positive equilibrium point, even if all non-isolated subsystems have the origin as the unique equilibrium point.*

Proof. The first part is easy and will be omitted.
Now suppose conditions 1 and 2 holds. We want to show that the whole network converges to a positive fixed point $\bar{\mathbf{x}} \in int(\mathbb{R}^n_+)$. Let us assume for simplicity that A is already in its normal form. We proceed iteratively starting from subsystem Σ_{h+1}. The states' $x_{\Sigma_{h+1}}$ of subsystem Σ_{h+1} is such that

$$\dot{x}_{\Sigma_{h+1}} = -\Delta_{h+1} x_{\Sigma_{h+1}} + \bar{A}_{h+1,h+1} \varphi_{\Sigma_{h+1}}(x) + b \tag{3.6}$$

where b has some positive elements and has the same dimension of $x_{\Sigma_{h+1}}$. This can be seen by looking at the normal form of A which states that there is at least one positive element in some block to the left of $\bar{A}_{h+1,h+1}$ indicated by $\star$. Since from condition 1 the first subsystems converge to the positive states, and their dynamics is isolated, we can focus on the dynamics when this convergence has happened. In this limit, the influence of the first subsystems on the $h+1$ one, modelled by b, can be assumed as a constant.
Let us take a permutation Π which splits b into $b^T = (b_1, b_2)$ such that $b_1 > 0$ and $b_2 = 0$. Applying the same transformation to the subsystem Σ_{h+1} we obtain $x^T_{\Sigma_{h+1}} = (x, y)$ such that

$$\begin{cases} \dot{x} = -\Delta_x x + f_x(x,y) + b_1 \\ \dot{y} = -\Delta_x y + f_y(x,y). \end{cases} \tag{3.7}$$

Even if the initial condition $x_{\Sigma h+1,0}$ of this subsystem was zero the x component becomes positive, because of the presence of b_1, while some component of y, for example y_k for some k, becomes positive because some y is influenced by at least one positive x_i. From the irreducibility of $\bar{A}_{h+1,h+1}$ we have that the whole state $x_{\Sigma_{h+1}}$ becomes positive at some $t > 0$. This means that the state can not have zero components.
Since adding positive constant does not change the increasing and strict subhomogeneity

property of a function, if we prove the existence of a positive fixed point for Σ_{h+1} then convergence is guaranteed for such system (see previous Theorem 2.2.3). We distinguish two cases:

1. Σ_{h+1} had a positive fixed point: Existence of a new positive fixed point is guaranteed by Tarski's Theorem 2.1.1, Theorem 2.2.1 and 2.4.2 since adding b does not change the spectral radius property of the subsystem's Jacobian matrix. We can thus conclude the existence of $a_1, a_2 \in \mathbb{R}^{|\Sigma_{h+1}|}$, $a_2 > a_1$ such that $f_{h+1}(a_1) > \Delta_{h+1}a_1$ and $f_{h+1}(a_2) < \Delta_{h+1}a_2$, with $f_{h+1}^T = (f_x^T + b^T, f_y^T)$.

2. 0 was the only equilibrium point for Σ_{h+1}: we have already proven that 0 is no longer an equilibrium point for this subsystem. If 0 was the only equilibrium point for Σ_{h+1} that means that $\rho\left(\frac{\partial f_{h+1}}{\partial x_{h+1}}(x)\right) < \delta_{min}$ for all x. This guarantee the existence of a_2 such that $f_{h+1}(a_2) < \Delta_{h+1}a_2$.
   If we manage to prove the existence of $a_1 > 0$ such that $f_{h+1}(a_1) > \Delta_{h+1}a_1$ then the proof is completed. With system (3.7) in mind, and since $f_x(x,y) + b_1$ is positive there exists a $\bar{x} > 0$ such that $f_x(\bar{x}, y) + b_1 > \Delta_{h+1}\bar{x}$ for all $y \geq 0$. With $\bar{x}$ fixed and since the subsystem is irreducible there exists an index j such that

$$\dot{y}_j = -\delta_j y_j + f_{yj}^x(\bar{x}) + f_{yj}^y(y) \tag{3.8}$$

   where f_{yj}^x and f_{yj}^x are sum functions only of $\bar{x}$ and y. Then $f_{yj}^x(\bar{x})$ is positive. This implies the existence of $\bar{y}_j$ such that $f_{yj}(\bar{x}, y_1, \ldots, \bar{y}_j, \ldots, y_m) > \delta_j y_j$. As stated, irreducibility implies that the same argumentations holds for all y components. This implies that $a_1 = (\bar{x}, \bar{y})$ is such that $f_{h+1}(a_1) > \Delta a_1$.

This completes the proof for Σ_{h+1}. So Σ_{h+1} always converges to a positive fixed point. By iterating this proof for $h+2, h+3, \ldots, k$ we conclude the proof that the whole systems (3.3) tends to $\bar{\mathbf{x}} \in int(\mathbb{R}_+^n)$. ■

3.2 Case studies

3.2.1 Irreducible example

The example here analysed is from Sompolinsky et al. [25]. The model consists of n localized continuous variables ("neurons") $\{S_i(t)\}, i = 1,\ldots,n$, where $-1 \leq S_i \leq 1$. Associated to each neuron, a local field h_i, $-\infty < h_i < +\infty$, is defined through the relationship $S_i(t) = \phi(h_i(t))$ where $\phi(x)$ is nonlinear gain function which defines the *input* (h_i)-*output*(S_i) characteristics of the neurons. In the biological context, h_i may be related to the membrane potential of the nerve cell and S_i to its electrical activity. The function $\phi(x)$ is assumed to have a sigmoid shape $\phi(\pm\infty) = \pm 1$, $\phi(-x) = -\phi(x)$. For our purposes we will think of the system as positive. The dynamics of the network is given by n coupled first-order differential equations ("circuit" equations)

$$\dot{h}_i = -h_i + \sum_{j=1}^{n} J_{ij} S_j = -h_i + \sum_{j=1}^{n} J_{ij}\phi(h_j). \tag{3.9}$$

Here J_{ij} is the synaptic efficacy which couples the output of the j^{th} neuron to the input of the i^{th} neuron, and $J_{ii} = 0$. For the sake of clarity let us start with an example in $\mathbb{R}^2$.
For $n = 2$ neurons, the system (2.2) with an hyperbolic tangent function, which is a sigmoid, becomes

$$\dot{x} = -x + J\varphi(x) = -x + J\begin{bmatrix} tanh(x_1) \\ tanh(x_2) \end{bmatrix} \tag{3.10}$$

cooperativity of the system means $J \geq 0$. Assuming for example

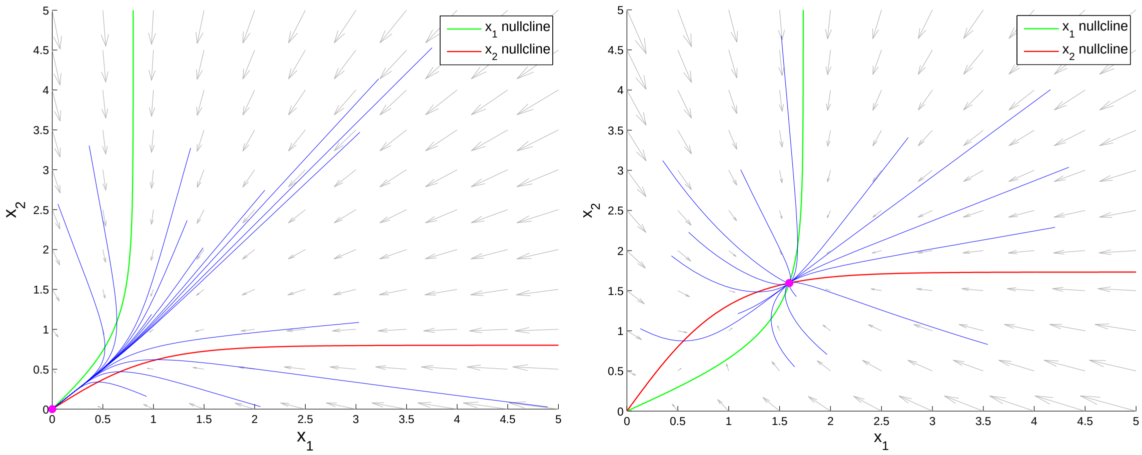
$$J = \begin{bmatrix} 0 & \delta_1 \\ \delta_2 & 0 \end{bmatrix}$$

which is irreducible and $\delta_{1,2} > 0$. From $\frac{\partial \varphi}{\partial x}(0) = I$, the spectral radius of the Jacobian matrix of $J\varphi(x)$ in 0 depends only on δ_1 and δ_2. This yields

$$\begin{cases} \dot{x}_1 = -x_1 + \delta_2 tanh(x_2) \\ \dot{x}_2 = -x_2 + \delta_1 tanh(x_1) \end{cases}$$

The nullclines of this systems are given by

$$\begin{cases} x_{1,null} = \delta_2 tanh(x_2) \\ x_{2,null} = \delta_1 tanh(x_1) \end{cases}$$



(a) Origin is GAS when $\delta_1 \delta_2 = 0.8 < 1$ (b) Positive point is GAS when $\delta_1 \delta_2 = 3 > 1$

Figure 3.1 2D irreducible example: Solution's trajectory and nullclines.

From Theorem 2.2.1 and Theorem 2.2.2 we can say that if $\rho(J) > 1$ then there exists a unique positive fixed point which is asymptotically stable in $\mathbb{R}^n_+ \backslash \{0\}$, while the origin becomes unstable. If $\rho(J) < 1$ the origin is the only equilibrium point and it is globally asymptotically stable. From $\rho(J) = \sqrt{\delta_1 \delta_2}$, we have a bifurcation at $\delta_1 \delta_2 = 1$ (see figure 3.1).

When $\delta_1 \delta_2 < 1$ the x_1-nullcline and the x_2-nullcline intersect only in one equilibrium. If $\delta_1 \delta_2 > 1$ the x_1-nullcline and the x_2-nullcline intersect in 3 equilibria: $\bar{x}_0 = 0, \bar{x}_1 \in \mathbb{R}^2_+$ and $\bar{x}_2 \in \mathbb{R}^2_-$. This property holds for arbitrarily large n.

Case $n = 500$

Let us consider the case where $n = 500$. From (3.9) we chose an irreducible and non-negative matrix J. As has been discussed in the previous sections the origin of the system passes from globally asymptotically stable to unstable whenever the spectral radius of J passes 1, this behaviour is depicted in figure 3.4. The trajectory's evolution are depicted in figure 3.2 if the initial condition x_0 is in the interior of $\mathbb{R}^n_+$, in figure 3.3 if $x_0 \in bd\left(\mathbb{R}^n_+\right)$.

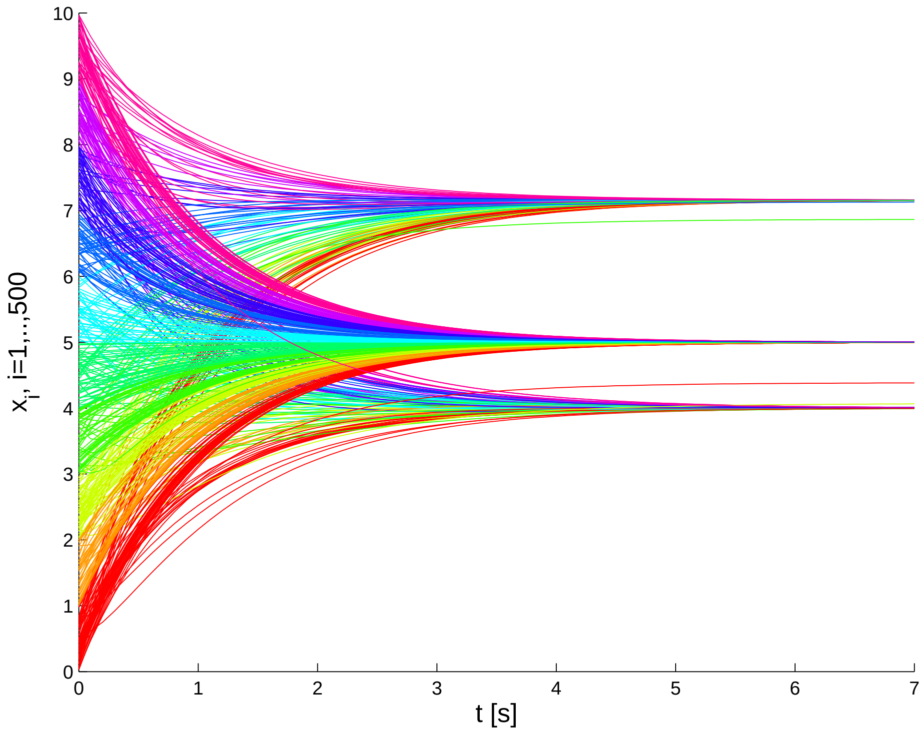


Figure 3.2 Trajectory's evolution with $\rho\left(\frac{\partial f}{\partial x}(0)\right) = 10$ and $x_0 \in int\left(\mathbb{R}^n_+\right)$.

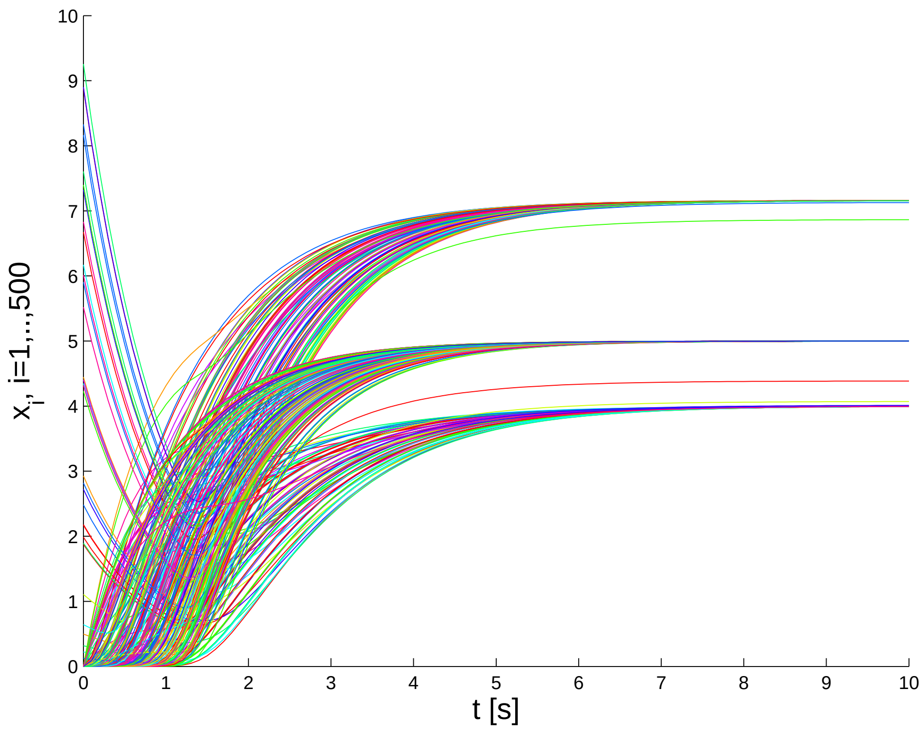


Figure 3.3 Trajectory's evolution with $\rho\left(\frac{\partial f}{\partial x}(0)\right) = 10$ and $x_0 \in bd\left(\mathbb{R}^n_+\right)$.

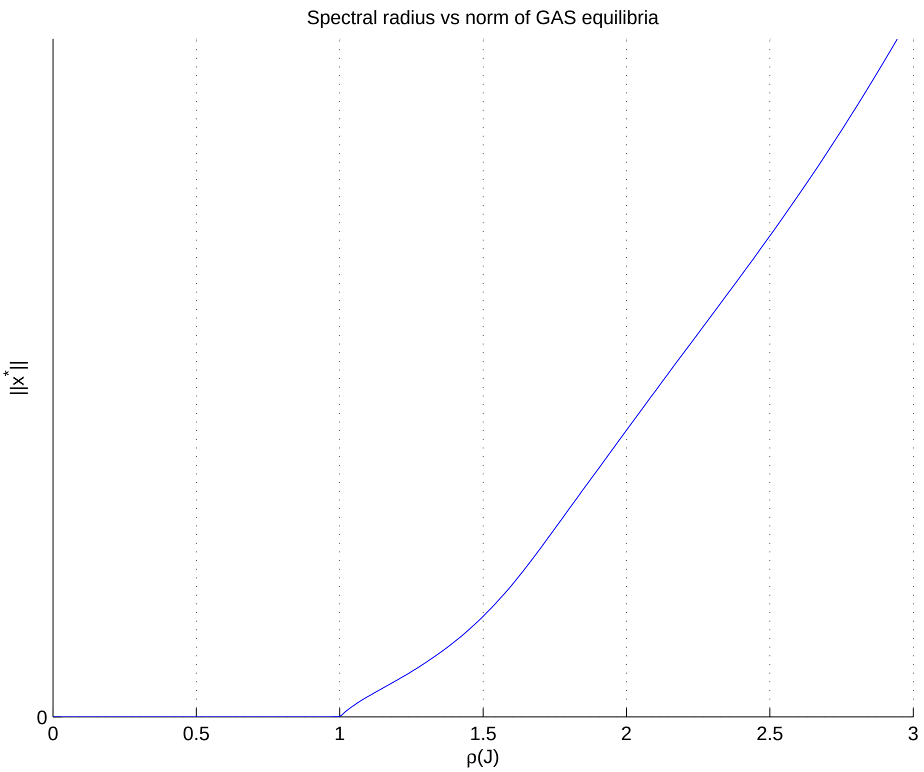


Figure 3.4 Irreducible neural network simulation in $\mathbb{R}^{500}$: Bifurcation at $\rho(J) = 1$.

3.2.2 Power Control in Wireless Networks

The power level chosen by transmitter i is denoted by p_i, v_i denotes the variance of thermal noise at the receiver i. The interference power at the i^{th} node, I_i, includes the interference from all the transmitter in the network and the thermal noise, and is defined in [14] in other to model the continuous-time power dynamics:

$$\frac{dp_i(t)}{dt} = k_i \left(-p_i(t) + \gamma_i \left(\sum_{j \neq i, j \in \mathscr{T}} \frac{g_{ji}}{g_{ii}} p_j(t) + \frac{v_i}{g_{ii}} \right) \right) \tag{3.11}$$

where $\mathscr{T}$ denotes the set of transmitters, $k_i \in \mathbb{R}_+,\ k_i > 0$ is the proportionality constant, g_{ji} denotes the channel gain on the link between transmitter j and receiver i and γ_i denotes the desired Signal-to-Interference-plus-Noise-Ratio. It is assumed that each transmitter i has knowledge of the interference at its receiver only,

$$I_i(p(t)) = \sum_{j \neq i, j \in \mathscr{T}} \frac{g_{ji}}{g_{ii}} p_j(t) + \frac{v_i}{g_{ii}}. \tag{3.12}$$

Defining $G = [g_{ij}]$ as the gain matrix, the whole equation can be written as

$$\dot{p}(t) = K \left(-p(t) + \Gamma \hat{G} p(t) + \eta \right) \tag{3.13}$$

where

$$\begin{aligned} K &= diag\{k_i\} \\ \Gamma &= diag\{\gamma_i\} \\ \eta &= diag(G)^{-1} \mathbb{1} \\ \hat{G} &= diag(G)^{-1}(G^T - diag(G)) \end{aligned} \tag{3.14}$$

therefore the interference function is

$$I(p(t)) = K\Gamma\hat{G}p(t) + K\eta \tag{3.15}$$

Thus equation (3.11) is similar to the system (2.2). Now $I(p) = Ap + b$ is strictly subhomogeneous of degree 1 if $b > 0$ thus conditions of Theorem 2.4.1 and Theorem 2.4.2 are satisfied if G is irreducible and $\rho(G) < 1$. Therefore under $\rho(G) < 1$ and irreducibility of G there exists a unique $p^* \in int(\mathbb{R}^n_+)$ such that $x(t, p_0) \to x^*$ for all $p_0 \in \mathbb{R}^n_+ \backslash \{0\}$.

3.2.3 Reducible example

Let us consider the following reducible example in $\mathbb{R}^{14}$:

$$A = \left[\begin{array}{cc|cc|c|ccc|ccc|cc|c}
0 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0.45 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.45 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 5 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0.4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0.25 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0
\end{array}\right].$$

As we can observe, the sub-matrices $A_{i,i}$ for $i = 1, \dots, 7$ are irreducible or 1×1 zero matrices. If we choose $\varphi_i(x_i) = \frac{x_i}{1+x_i}$, $i = 1, \dots, 7$ then the spectral radius at the origin, i.e. $\rho\left(\frac{\partial}{\partial x} A\varphi(0)\right)$ of each subsystems are as reported in table 3.1.

Table 3.1 Degradation rate and spectral radius of each subsystem

Subsystem	Properties	
	Degradation rate δ	Spectral radius ρ
$A_{1,1}$	0.1	0.4
$A_{2,2}$	0.1	0.45
$A_{3,3}$	0.5	0
$A_{4,4}$	0.5	0.25
$A_{5,5}$	0.5	0.4
$A_{6,6}$	0.5	0.25
$A_{7,7}$	0.5	0

From table 3.1, since the isolated systems Σ_1 and Σ_2 has degradation rate $\delta_i = 0.1$ for $i \in \Sigma_1 \cup \Sigma_2$ then they converge to a positive fixed point.

The non-isolated systems Σ_3, Σ_4, Σ_5, Σ_6 and Σ_7 has $\delta_i = 0.5$ for $i \in \bigcup_{i=3}^{7} \Sigma_i$ then they converge to the origin, which is the only equilibrium point. The conditions of Theorem 3.1.1 are satisfied thus the whole interconnected system converges to a positive fixed point if x_0 is for example $x_0 = (1,0,1,0,0,0,0,0,0,0,0,0,0,0)^T$

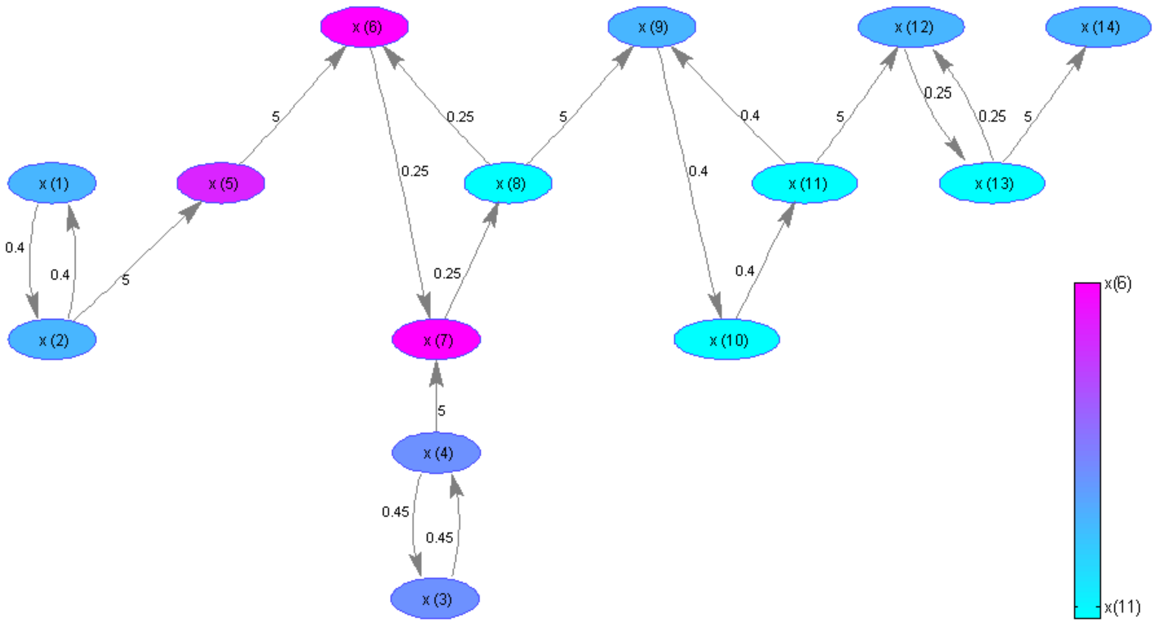


Figure 3.5 Reducible network simulation

From Figure 3.5 we can see that the network converges to a positive fixed point $\bar{\mathbf{x}} \in \mathbb{R}^{14}$. In fact the smallest component of $\bar{\mathbf{x}}$, which is $\bar{\mathbf{x}}_{11}$, is greater than zero.

Chapter 4

Conclusions

In this thesis we have analysed and fully characterized the stability properties of positive nonlinear systems with degradation terms and in which the trajectory's evolution is monotone.

First of all we stated and proved all necessary results for the case of concave system dynamics, leading to a main self-contained theorem that guarantees existence and uniqueness of an equilibrium point in the interior of $\mathbb{R}^n_+$. These results were later extended to reducible systems. Furthermore, we made a comparison to standard interference function and used the scalability property, i.e. subhomogeneity, to extend our studies. This led to a more general theorem that has as a sub case the theorem for systems with concave dynamics. In fact, concavity implies subhomogeneity of degree 1.

A very import result demonstrated both for concave and subhomogeneous vector field is the lemma that states that the spectral radius of the Jacobian matrix of f calculated in the positive equilibrium point must be less that one. This implies that the spectral abscissa of the Jacobian matrix of the whole system calculated in this point is less than zero, thus Hurwitz. For a generic nonlinear systems it is a sufficient condition to guarantee local asymptotic stability in a neighbourhood but, under the assumptions of our main theorems, asymptotic stability in the entire interior of $\mathbb{R}^n_+$ is guaranteed.

4.1 Further developments

The class of subhomogeneous vector fields considered in this manuscript can be broaden. For example the degradation term can be merged into the vector field f, i.e. ignored, and then some conditions on the system's vector field from our main theorems should be changed. One possible approach could be finding some sufficient conditions so that subhomogeneous vector field of degree 1 satisfies Schauder's theorem, thus guaranteeing the existence of a fixed point. Under assumptions of cooperativity uniqueness and convergence will still be guaranteed.

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Appendix A

Simulation framework

Simulation $\mathbb{R}^{500}$

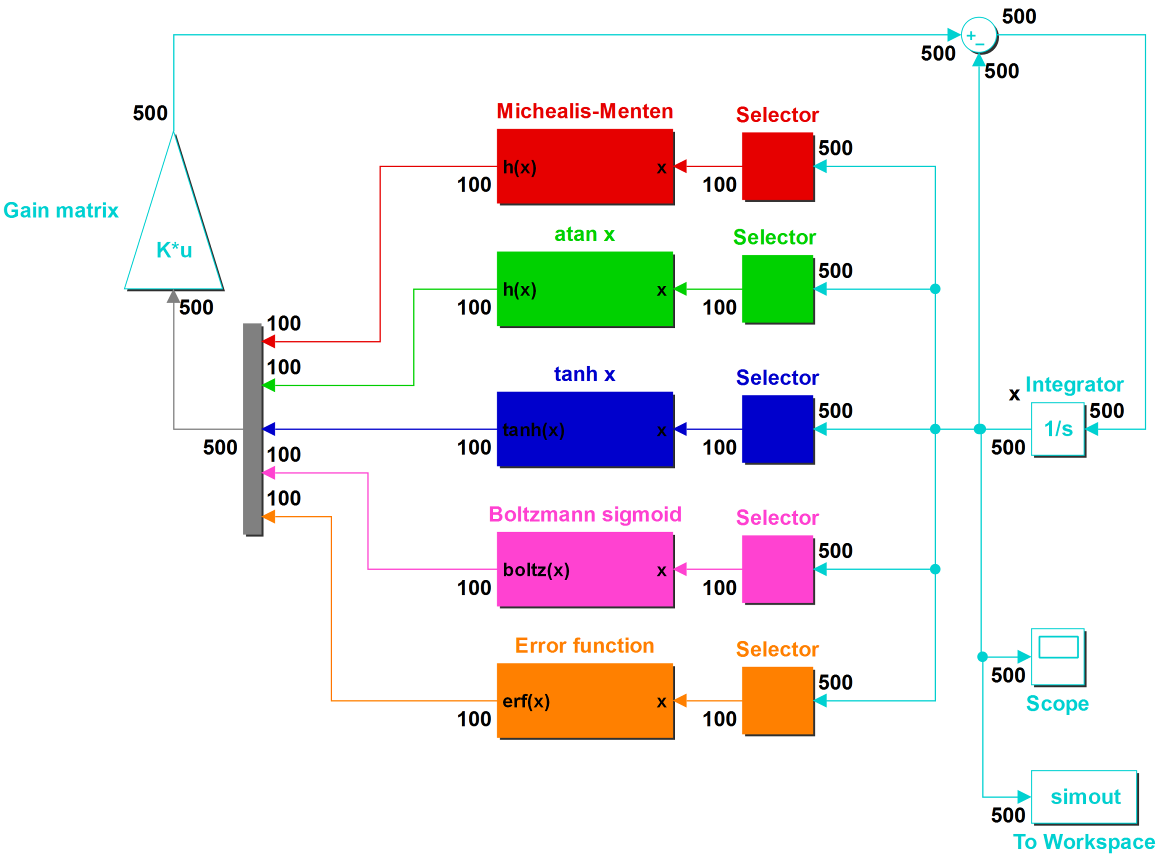

Figure A.1 Simulink model for simulation of case $\mathbb{R}^{500}$

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