Slotine • Li

APPLIED NONLINEAR CONTROL

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To Our Parents

Contents

Preface

In recent years, the availability of powerful low-cost microprocessors has spurred great advances in the theory and applications of nonlinear control. In terms of theory, major strides have been made in the areas of feedback linearization, sliding control, and nonlinear adaptation techniques. In terms of applications, many practical nonlinear control systems have been developed, ranging from digital "fly-by-wire" flight control systems for aircraft, to "drive-by-wire" automobiles, to advanced robotic and space systems. As a result, the subject of nonlinear control is occupying an increasingly important place in automatic control engineering, and has become a necessary part of the fundamental background of control engineers.

This book, based on a course developed at MIT, is intended as a textbook for senior and graduate students, and as a self-study book for practicing engineers. Its objective is to present the fundamental results of modern nonlinear control while keeping the mathematical complexity to a minimum, and to demonstrate their use and implications in the design of practical nonlinear control systems. Although a major motivation of this book is to detail the many recent developments in nonlinear control, classical techniques such as phase plane analysis and the describing function method are also treated, because of their continued practical importance.

In order to achieve our fundamental objective, we have tried to bring the following features to this book:

- **Readability:** Particular attention is paid to the readability of the book by carefully organizing the concepts, intuitively interpreting the major results, and selectively using the mathematical tools. The readers are only assumed to have had one introductory control course. No mathematical background beyond ordinary differential equations and elementary matrix algebra is required. For each new result, interpretation is emphasized rather than mathematics. For each major result, we try to ask and answer the following key questions: What does the result intuitively and physically mean? How can it be applied to practical problems? What is its relationship to other theorems? All major concepts and results are demonstrated by examples. We believe that learning and generalization from examples are crucial for proficiency in applying any theoretical result.

- **Practicality:** The choice and emphasis of materials is guided by the basic objective of making an engineer or student capable of dealing with practical control problems in industry. Some results of mostly theoretical interest are not included. The selected materials, in one way or another, are intended to allow readers to gain insights into the solution of real problems.

• **Comprehensiveness:** The book contains both classical materials, such as Lyapunov analysis and describing function techniques, and more modern topics such as feedback linearization, adaptive control, and sliding control. To facilitate digestion, asterisks are used to indicate sections which, given their relative complexity, can be safely skipped in a first reading.

• **Currentness:** In the past few years, a number of major results have been obtained in nonlinear control, particularly in nonlinear control system design and in robotics. It is one of the objectives of this book to present these new and important developments, and their implications, in a clear, easily understandable fashion. The book can thus be used as a reference and a guide to the active literature in these fields.

The book is divided into two major parts. Chapters 2-5 present the major *analytical* tools that can be used to study a nonlinear system, while chapters 6-9 treat the major nonlinear controller *design* techniques. Each chapter is supplied with exercises, allowing the reader to further explore specific aspects of the material discussed. A detailed index and a bibliography are provided at the end of the book.

The material included exceeds what can be taught in one semester or self-learned in a short period. The book can be studied in many ways, according to the particular interests of the reader or the instructor. We recommend that a first reading include a detailed study of chapter 3 (basic Lyapunov theory), sections 4.5-4.7 (Barbalat's lemma and passivity tools), section 6.1 and parts of sections 6.2-6.4 (feedback linearization), chapter 7 (sliding control), sections 8.1-8.3 and 8.5 (adaptive control of linear and nonlinear systems), and chapter 9 (control of multi-input physical systems). Conversely, sections denoted with an asterisk can be skipped in a first reading.

Many colleagues, students, and friends greatly contributed to this book through stimulating discussions and judicious suggestions. Karl Hedrick provided us with continued enthusiasm and encouragement, and with many valuable comments and suggestions. Discussions with Karl Aström and Semyon Meerkov helped us better define the tone of the book and its mathematical level. Harry Asada, Jo Bentsman, Marika DiBenedetto, Olav Egeland, Neville Hogan, Marija Ilic, Lars Nielsen, Ken Salisbury, Sajhendra Singh, Mark Spong, David Wormley, and Dana Yoerger provided many useful suggestions and much moral support. Barbara Hove created most of the nicer drawings in the book; Günter Niemeyer's expertise and energy was invaluable in setting up the computing and word processing environments; Hyun Yang greatly helped with the computer simulations; all three provided us with extensive technical and editorial comments. The book also greatly benefited from the interest and enthusiasm of many students who took the course at MIT.

Partial summer support for the first author towards the development of the book was provided by Gordon Funds. Finally, the energy and professionalism of Tim Bozik and Jennifer Wenzel at Prentice-Hall were very effective and highly appreciated.

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Weiping Li

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Chapter 1
Introduction

The subject of nonlinear control deals with the analysis and the design of nonlinear control systems, *i.e.*, of control systems containing at least one nonlinear component. In the analysis, a nonlinear closed-loop system is assumed to have been designed, and we wish to determine the characteristics of the system's behavior. In the design, we are given a nonlinear plant to be controlled and some specifications of closed-loop system behavior, and our task is to construct a controller so that the closed loop system meets the desired characteristics. In practice, of course, the issues of design and analysis are intertwined, because the design of a nonlinear control system usually involves an iterative process of analysis and design.

This introductory chapter provides the background for the specific analysis and design methods to be discussed in the later chapters. Section 1.1 explains the motivations for embarking on a study of nonlinear control. The unique and rich behaviors exhibited by nonlinear systems are discussed in section 1.2. Finally, section 1.3 gives an overview of the organization of the book.

1.1 Why Nonlinear Control ?

Linear control is a mature subject with a variety of powerful methods and a long history of successful industrial applications. Thus, it is natural for one to wonder why so many researchers and designers, from such broad areas as aircraft and spacecraft control, robotics, process control, and biomedical engineering, have recently showed an active interest in the development and applications of nonlinear control methodologies. Many reasons can be cited for this interest:

• **Improvement of existing control systems:** Linear control methods rely on the key assumption of small range operation for the linear model to be valid. When the required operation range is large, a linear controller is likely to perform very poorly or to be unstable, because the nonlinearities in the system cannot be properly compensated for. Nonlinear controllers, on the other hand, may handle the nonlinearities in large range operation directly. This point is easily demonstrated in robot motion control problems. When a linear controller is used to control robot motion, it neglects the nonlinear forces associated with the motion of the robot links. The controller's accuracy thus quickly degrades as the speed of motion increases, because many of the dynamic forces involved, such as Coriolis and centripetal forces, vary as the square of the speed. Therefore, in order to achieve a pre-specified accuracy in robot tasks such as pick-and-place, arc welding and laser cutting, the speed of robot motion, and thus productivity, has to be kept low. On the other hand, a conceptually simple nonlinear controller, commonly called computed torque controller, can fully compensate the nonlinear forces in the robot motion and lead to high accuracy control for a very large range of robot speeds and a large workspace.

• **Analysis of hard nonlinearities:** Another assumption of linear control is that the system model is indeed linearizable. However, in control systems there are many nonlinearities whose discontinuous nature does not allow linear approximation. These so-called "hard nonlinearities" include Coulomb friction, saturation, dead-zones, backlash, and hysteresis, and are often found in control engineering. Their effects cannot be derived from linear methods, and nonlinear analysis techniques must be developed to predict a system's performance in the presence of these inherent nonlinearities. Because such nonlinearities frequently cause undesirable behavior of the control systems, such as instabilities or spurious limit cycles, their effects must be predicted and properly compensated for.

• **Dealing with model uncertainties:** In designing linear controllers, it is usually necessary to assume that the parameters of the system model are reasonably well known. However, many control problems involve uncertainties in the model parameters. This may be due to a slow time variation of the parameters (*e.g.*, of ambient air pressure during an aircraft flight), or to an abrupt change in parameters (*e.g.*, in the inertial parameters of a robot when a new object is grasped). A linear controller based on inaccurate or obsolete values of the model parameters may exhibit significant performance degradation or even instability. Nonlinearities can be intentionally introduced into the controller part of a control system so that model uncertainties can be tolerated. Two classes of nonlinear controllers for this purpose are robust controllers and adaptive controllers.

• **Design Simplicity:** Good nonlinear control designs may be simpler and more intuitive than their linear counterparts. This *a priori* paradoxical result comes from the fact that nonlinear controller designs are often deeply rooted in the physics of the plants. To take a very simple example, consider a swinging pendulum attached to a hinge, in the vertical plane. Starting from some arbitrary initial angle, the pendulum will oscillate and progressively stop along the vertical. Although the pendulum's behavior could be analyzed close to equilibrium by linearizing the system, physically its stability has very little to do with the eigenvalues of some linearized system matrix: it comes from the fact that the total mechanical energy of the system is progressively dissipated by various friction forces (*e.g.*, at the hinge), so that the pendulum comes to rest at a position of minimal energy.

There may be other related or unrelated reasons to use nonlinear control techniques, such as cost and performance optimality. In industrial settings, ad-hoc extensions of linear techniques to control advanced machines with significant nonlinearities may result in unduly costly and lengthy development periods, where the control code comes with little stability or performance guarantees and is extremely hard to transport to similar but different applications. Linear control may require high quality actuators and sensors to produce linear behavior in the specified operation range, while nonlinear control may permit the use of less expensive components with nonlinear characteristics. As for performance optimality, we can cite bang-bang type controllers, which can produce fast response, but are inherently nonlinear.

Thus, the subject of nonlinear control is an important area of automatic control. Learning basic techniques of nonlinear control analysis and design can significantly enhance the ability of a control engineer to deal with practical control problems effectively. It also provides a sharper understanding of the real world, which is inherently nonlinear. In the past, the application of nonlinear control methods had been limited by the computational difficulty associated with nonlinear control design and analysis. In recent years, however, advances in computer technology have greatly relieved this problem. Therefore, there is currently considerable enthusiasm for the research and application of nonlinear control methods. The topic of nonlinear control design for large range operation has attracted particular attention because, on the one hand, the advent of powerful microprocessors has made the implementation of nonlinear controllers a relatively simple matter, and, on the other hand, modern technology, such as high-speed high-accuracy robots or high-performance aircrafts, is demanding control systems with much more stringent design specifications. Nonlinear control occupies an increasingly conspicuous position in control engineering, as reflected by the ever-increasing number of papers and reports on nonlinear control research and applications.

1.2 Nonlinear System Behavior

Physical systems are inherently nonlinear. Thus, all control systems are nonlinear to a certain extent. Nonlinear control systems can be described by nonlinear differential equations. However, if the operating range of a control system is small, and if the involved nonlinearities are smooth, then the control system may be reasonably approximated by a linearized system, whose dynamics is described by a set of linear differential equations.

NONLINEARITIES

Nonlinearities can be classified as *inherent (natural)* and *intentional (artificial)*. Inherent nonlinearities are those which naturally come with the system's hardware and motion. Examples of inherent nonlinearities include centripetal forces in rotational motion, and Coulomb friction between contacting surfaces. Usually, such nonlinearities have undesirable effects, and control systems have to properly compensate for them. Intentional nonlinearities, on the other hand, are artificially introduced by the designer. Nonlinear control laws, such as adaptive control laws and bang-bang optimal control laws, are typical examples of intentional nonlinearities.

Nonlinearities can also be classified in terms of their mathematical properties, as *continuous* and *discontinuous*. Because discontinuous nonlinearities cannot be locally approximated by linear functions, they are also called "hard" nonlinearities. Hard nonlinearities (such as, *e.g.*, backlash, hysteresis, or stiction) are commonly found in control systems, both in small range operation and large range operation. Whether a system in small range operation should be regarded as nonlinear or linear depends on the magnitude of the hard nonlinearities and on the extent of their effects on the system performance. A detailed discussion of hard nonlinearities is provided in section 5.2.

LINEAR SYSTEMS

Linear control theory has been predominantly concerned with the study of linear time-invariant (LTI) control systems, of the form

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} \tag{1.1}$$

with $\mathbf{x}$ being a vector of states and $\mathbf{A}$ being the system matrix. LTI systems have quite simple properties, such as

- a linear system has a *unique equilibrium point* if $\mathbf{A}$ is nonsingular;

- the equilibrium point is stable if all eigenvalues of $\mathbf{A}$ have negative real parts, *regardless of initial conditions*;

- the transient response of a linear system is composed of the natural modes of the system, and the general solution can be solved analytically;

- in the presence of an external input $\mathbf{u}(t)$, *i.e.*, with

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} \tag{1.2}$$

the system response has a number of interesting properties. First, it satisfies the *principle of superposition*. Second, the asymptotic stability of the system (1.1) implies bounded-input bounded-output stability in the presence of $\mathbf{u}$. Third, a sinusoidal input leads to a sinusoidal output of the same frequency.

AN EXAMPLE OF NONLINEAR SYSTEM BEHAVIOR

The behavior of nonlinear systems, however, is much more complex. Due to the lack of linearity and of the associated superposition property, nonlinear systems respond to external inputs quite differently from linear systems, as the following example illustrates.

Example 1.1: A simplified model of the motion of an underwater vehicle can be written

$$\dot{v} + |v|\,v = u \tag{1.3}$$

where v is the vehicle velocity and u is the control input (the thrust provided by a propeller). The nonlinearity $|v|\,v$ corresponds to a typical "square-law" drag.

Assume that we apply a unit step input in thrust u, followed 5 seconds later by a negative unit step input. The system response is plotted in Figure 1.1. We see that the system settles much faster in response to the positive unit step than it does in response to the subsequent negative unit step. Intuitively, this can be interpreted as reflecting the fact that the "apparent damping" coefficient $|v|$ is larger at high speeds than at low speeds.

Assume now that we repeat the same experiment but with larger steps, of amplitude 10. Predictably, the difference between the settling times in response to the positive and negative steps is even more marked (Figure 1.2). Furthermore, the settling speed v_s in response to the first step is *not* 10 times that obtained in response to the first unit step in the first experiment, as it would be in a linear system. This can again be understood intuitively, by writing that

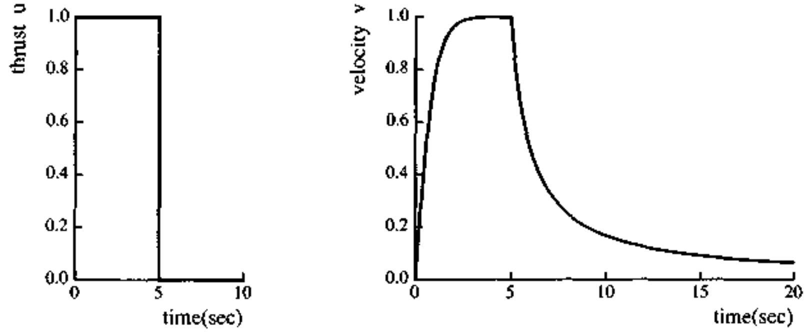


Figure 1.1 : Response of system (1.3) to unit steps

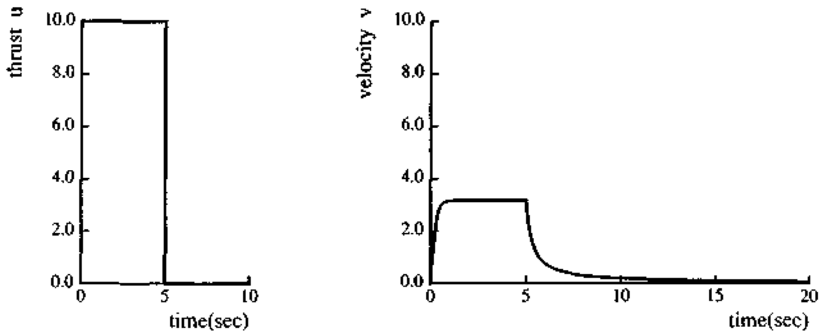


Figure 1.2 : Response of system (1.3) to steps of amplitude 10

$$u = 1 \quad => \quad 0 + |v_s|\, v_s = 1 \quad => \quad v_s = 1$$

$$u = 10 \quad => \quad 0 + |v_s|\, v_s = 10 \quad => \quad v_s = \sqrt{10} \approx 3.2$$

Carefully understanding and effectively controlling this nonlinear behavior is particularly important if the vehicle is to move in a large dynamic range and change speeds continually, as is typical of industrial remotely-operated underwater vehicles (R.O.V.'s). □

SOME COMMON NONLINEAR SYSTEM BEHAVIORS

Let us now discuss some common nonlinear system properties, so as to familiarize ourselves with the complex behavior of nonlinear systems and provide a useful background for our study in the rest of the book.

Multiple Equilibrium Points

Nonlinear systems frequently have more than one equilibrium point (an equilibrium point is a point where the system can stay forever without moving, as we shall formalize later). This can be seen by the following simple example.

Example 1.2: A first-order system

Consider the first order system

$$\dot{x} = -x + x^2 \tag{1.4}$$

with initial condition $x(0) = x_o$. Its linearization is

$$\dot{x} = -x \tag{1.5}$$

The solution of this linear equation is $x(t) = x_o\, e^{-t}$. It is plotted in Figure 1.3(a) for various initial conditions. The linearized system clearly has a unique equilibrium point at $x = 0$.

By contrast, integrating equation $dx/(-x + x^2) = dt$, the actual response of the nonlinear dynamics (1.4) can be found to be

$$x(t) = \frac{x_o\, e^{-t}}{1 - x_o + x_o e^{-t}}$$

This response is plotted in Figure 1.3(b) for various initial conditions. The system has two equilibrium points, $x = 0$ and $x = 1$, and its qualitative behavior strongly depends on its initial condition. □

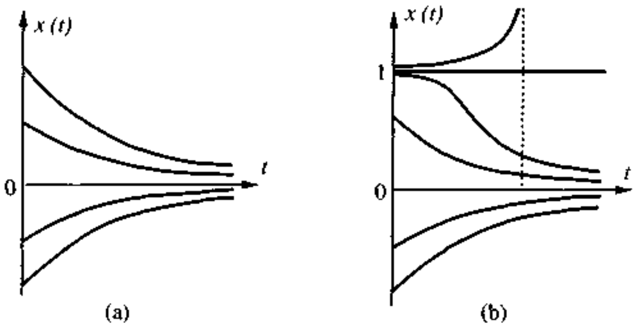


Figure 1.3 : Responses of the linearized system (a) and the nonlinear system (b)

The issue of motion stability can also be discussed with the aid of the above example. For the linearized system, stability is seen by noting that for *any* initial condition, the motion always converges to the equilibrium point $x = 0$. However, consider now the actual nonlinear system. While motions starting with $x_o < 1$ will indeed converge to the equilibrium point $x = 0$, those starting with $x_o > 1$ will go to infinity (actually in finite time, a phenomenon known as finite escape time). This means that the stability of nonlinear systems may depend on initial conditions.

In the presence of a bounded external input, stability may also be dependent on the input value. This input dependence is highlighted by the so-called bilinear system

$$\dot{x} = x\,u$$

If the input u is chosen to be -1, then the state x converges to 0. If $u = 1$, then $|x|$ tends to infinity.

Limit Cycles

Nonlinear systems can display oscillations of fixed amplitude and fixed period without external excitation. These oscillations are called limit cycles, or self-excited oscillations. This important phenomenon can be simply illustrated by a famous oscillator dynamics, first studied in the 1920's by the Dutch electrical engineer Balthasar Van der Pol.

Example 1.3: Van der Pol Equation

The second-order nonlinear differential equation

$$m\ddot{x} + 2c(x^2 - 1)\dot{x} + kx = 0 \qquad (1.6)$$

where m, c and k are positive constants, is the famous Van der Pol equation. It can be regarded as describing a mass-spring-damper system with a position-dependent damping coefficient $2c(x^2 - 1)$ (or, equivalently, an RLC electrical circuit with a nonlinear resistor). For large values of x, the damping coefficient is positive and the damper removes energy from the system. This implies that the system motion has a convergent tendency. However, for small values of x, the damping coefficient is negative and the damper adds energy into the system. This suggests that the system motion has a divergent tendency. Therefore, because the nonlinear damping varies with x, the system motion can neither grow unboundedly nor decay to zero. Instead, it displays a sustained oscillation independent of initial conditions, as illustrated in Figure 1.4. This so-called limit cycle is sustained by periodically releasing energy into and absorbing energy from the environment, through the damping term. This is in contrast with the case of a conservative mass-spring system, which does not exchange energy with its environment during its vibration. □

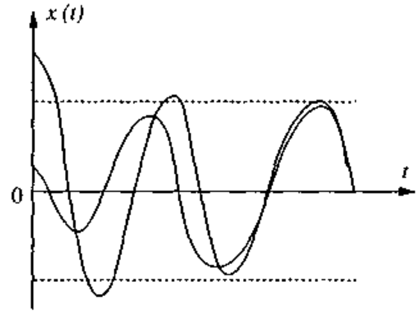


Figure 1.4 : Responses of the Van der Pol oscillator

Of course, sustained oscillations can also be found in linear systems, in the case of marginally stable linear systems (such as a mass-spring system without damping) or in the response to sinusoidal inputs. However, limit cycles in nonlinear systems are different from linear oscillations in a number of fundamental aspects. First, the amplitude of the self-sustained excitation is independent of the initial condition, as seen in Figure 1.2, while the oscillation of a marginally stable linear system has its amplitude determined by its initial conditions. Second, marginally stable linear systems are very sensitive to changes in system parameters (with a slight change capable of leading either to stable convergence or to instability), while limit cycles are not easily affected by parameter changes.

Limit cycles represent an important phenomenon in nonlinear systems. They can be found in many areas of enginering and nature. Aircraft wing fluttering, a limit cycle caused by the interaction of aerodynamic forces and structural vibrations, is frequently encountered and is sometimes dangerous. The hopping motion of a legged robot is another instance of a limit cycle. Limit cycles also occur in electrical circuits, *e.g.*, in laboratory electronic oscillators. As one can see from these examples, limit cycles can be undesirable in some cases, but desirable in other cases. An engineer has to know how to eliminate them when they are undesirable, and conversely how to generate or amplify them when they are desirable. To do this, however, requires an understanding of the properties of limit cycles and a familiarity with the tools for manipulating them.

Bifurcations

As the parameters of nonlinear dynamic systems are changed, the stability of the equilibrium point can change (as it does in linear systems) and so can the number of equilibrium points. Values of these parameters at which the qualitative nature of the system's motion changes are known as *critical* or *bifurcation* values. The phenomenon of bifurcation, *i.e.*, quantitative change of parameters leading to qualitative change of system properties, is the topic of bifurcation theory.

For instance, the smoke rising from an incense stick (smokestacks and cigarettes are old-fashioned) first accelerates upwards (because it is lighter than the ambient air), but beyond some critical velocity breaks into swirls. More prosaically, let us consider the system described by the so-called undamped Duffing equation

$$\ddot{x} + \alpha x + x^3 = 0$$

(the damped Duffing equation is $\ddot{x} + c\dot{x} + \alpha x + \beta x^3 = 0$, which may represent a mass-damper-spring system with a hardening spring). We can plot the equilibrium points as a function of the parameter α. As α varies from positive to negative, one equilibrium point splits into *three* points ($x_e = 0, \sqrt{\alpha}, -\sqrt{\alpha}$), as shown in Figure 1.5(a). This represents a qualitative change in the dynamics and thus $\alpha = 0$ is a critical bifurcation value. This kind for bifurcation is known as a *pitchfork*, due to the shape of the equilibrium point plot in Figure 1.5(a).

Another kind of bifurcation involves the emergence of limit cycles as parameters are changed. In this case, a pair of complex conjugate eigenvalues $p_1 = \gamma + j\omega$, $p_2 = \gamma - j\omega$ cross from the left-half plane into the right-half plane, and the response of the unstable system diverges to a limit cycle. Figure 1.5(b) depicts the change of typical system state trajectories (states are x and $\dot{x}$) as the parameter α is varied. This type of bifurcation is called a Hopf bifurcation.

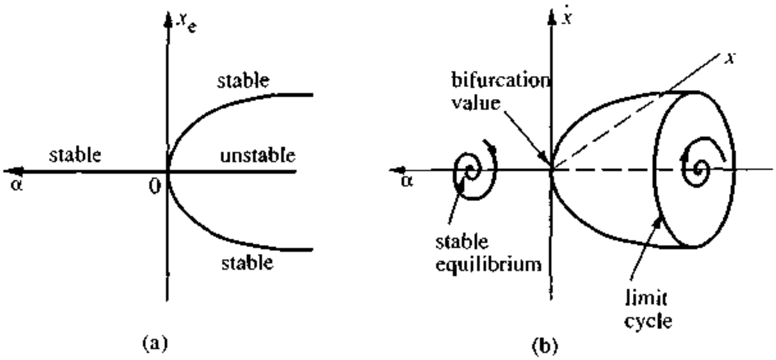


Figure 1.5 : (a) a pitchfork bifurcation; (b) a Hopf bifurcation

Chaos

For stable linear systems, small differences in initial conditions can only cause small differences in output. Nonlinear systems, however, can display a phenomenon called *chaos*, by which we mean that the system output is extremely sensitive to initial conditions. The essential feature of chaos is the unpredictability of the system output. Even if we have an exact model of a nonlinear system and an extremely accurate computer, the system's response in the long-run still cannot be well predicted.

Chaos must be distinguished from random motion. In random motion, the system model or input contain uncertainty and, as a result, the time variation of the output cannot be predicted exactly (only statistical measures are available). In chaotic motion, on the other hand, the involved problem is deterministic, and there is little uncertainty in system model, input, or initial conditions.

As an example of chaotic behavior, let us consider the simple nonlinear system

$$\ddot{x} + 0.1\,\dot{x} + x^5 = 6\,\sin t$$

which may represent a lightly-damped, sinusoidally forced mechanical structure undergoing large elastic deflections. Figure 1.6 shows the responses of the system corresponding to two almost identical initial conditions, namely $x(0) = 2, \dot{x}(0) = 3$ (thick line) and $x(0) = 2.01, \dot{x}(0) = 3.01$ (thin line). Due to the presence of the strong nonlinearity in x^5, the two responses are radically different after some time.

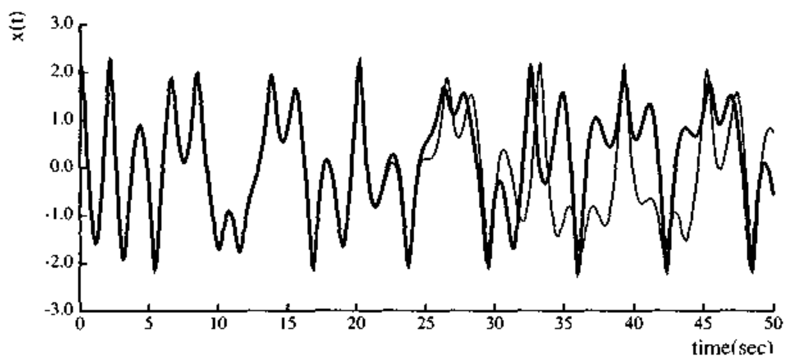


Figure 1.6 : Chaotic behavior of a nonlinear system

Chaotic phenomena can be observed in many physical systems. The most commonly seen physical problem is turbulence in fluid mechanics (such as the swirls of our incense stick). Atmospheric dynamics also display clear chaotic behavior, thus making long-term weather prediction impossible. Some mechanical and electrical systems known to exhibit chaotic vibrations include buckled elastic structures, mechanical systems with play or backlash, systems with aeroelastic dynamics, wheel-rail dynamics in railway systems, and, of course, feedback control devices.

Chaos occurs mostly in *strongly* nonlinear systems. This implies that, for a given system, if the initial condition or the external input cause the system to operate in a highly nonlinear region, it increases the possibility of generating chaos. Chaos cannot occur in linear systems. Corresponding to a sinusoidal input of arbitrary magnitude, the linear system response is always a sinusoid of the same frequency. By contrast, the output of a given nonlinear system may display sinusoidal, periodic, or chaotic behaviors, depending on the initial condition and the input magnitude.

In the context of feedback control, it is of course of interest to know when a nonlinear system will get into a chaotic mode (so as to avoid it) and, in case it does, how to recover from it. Such problems are the object of active research.

Other behaviors

Other interesting types of behavior, such as jump resonance, subharmonic generation, asynchronous quenching, and frequency-amplitude dependence of free vibrations, can also occur and become important in some system studies. However, the above description should provide ample evidence that nonlinear systems can have considerably richer and more complex behavior than linear systems.

1.3 An Overview of the Book

Because nonlinear systems can have much richer and more complex behaviors than linear systems, their analysis is much more difficult. Mathematically, this is reflected in two aspects. First, nonlinear equations, unlike linear ones, cannot in general be solved analytically, and therefore a complete understanding of the behavior of a nonlinear system is very difficult. Second, powerful mathematical tools like Laplace and Fourier transforms do not apply to nonlinear systems.

As a result, there are no systematic tools for predicting the behavior of nonlinear systems, nor are there systematic procedures for designing nonlinear control systems. Instead, there is a rich inventory of powerful analysis and design tools, each best applicable to particular classes of nonlinear control problems. It is the objective of this book to present these various tools, with particular emphasis on their powers and limitations, and on how they can be effectively combined.

This book is divided into two major parts. Part I (chapters 2-5) presents the major *analytical* tools that can be used to study a nonlinear system. Part II (chapters 6-9) discusses the major nonlinear controller *design* techniques. Each part starts with a short introduction providing the background for the main issues and techniques to be discussed.

In chapter 2, we further familiarize ourselves with some basic nonlinear system behaviors, by studying second-order systems using the simple graphical tools provided by so-called phase plane analysis. Chapter 3 introduces the most fundamental analysis tool to be used in this book, namely the concept of a Lyapunov function and its use in nonlinear stability analysis. Chapter 4 studies selected advanced topics in stability analysis. Chapter 5 discusses an approximate nonlinear system analysis method, the describing function method, which aims at extending to nonlinear systems some of the desirable and intuitive properties of linear frequency response analysis.

The basic idea of chapter 6 is to study under what conditions the dynamics of a nonlinear system can be algebraically transformed in that of a linear system, on which linear control design techniques can in turn be applied. Chapters 7 and 8 then study how to reduce or practically eliminate the effects of model uncertainties on the stability and performance of feedback controllers for linear or nonlinear systems, using so-called robust and adaptive approaches. Finally, chapter 9 extensively discusses the use of known physical properties to simplify and enhance the design of controllers for complex multi-input nonlinear systems.

The book concentrates on nonlinear systems represented in continuous-time form. Even though most control systems are implemented digitally, nonlinear physical systems are continuous in nature and are hard to meaningfully discretize, while digital control systems may be treated as continuous-time systems in analysis and design if high sampling rates are used. Given the availability of cheap computation, the most common practical case when it may be advantageous to consider sampling explicitly is when *measurements* are sparse, as *e.g.*, in the case of underwater vehicles using acoustic navigation. Some practical issues involved in the digital implementation of controllers designed from continuous-time formulations are discussed in the introduction to Part II.

1.4 Notes and References

Detailed discussions of bifurcations and chaos can be found, *e.g.*, in [Guckenheimer and Holmes, 1983] and in [Thompson and Stewart, 1986], from which the example of Figure 1.6 is adapted.

Part I
Nonlinear Systems Analysis

The objective of this part is to present various tools available for analyzing nonlinear control systems. The study of these nonlinear analysis techniques is important for a number of reasons. First, theoretical analysis is usually the least expensive way of exploring a system's characteristics. Second, simulation, though very important in nonlinear control, has to be guided by theory. Blind simulation of nonlinear systems is likely to produce few results or misleading results. This is especially true given the great richness of behavior that nonlinear systems can exhibit, depending on initial conditions and inputs. Third, the design of nonlinear controllers is always based on analysis techniques. Since design methods are usually based on analysis methods, it is almost impossible to master the design methods without first studying the analysis tools. Furthermore, analysis tools also allow us to assess control designs after they have been made, and, in case of inadequate performance, they may also suggest directions of modifying the control designs.

It should not come as a surprise that no universal technique has been devised for the analysis of all nonlinear control systems. In linear control, one can analyze a system in the time domain or in the frequency domain. However, for nonlinear control systems, none of these standard approaches can be used, since direct solution of nonlinear differential equations is generally impossible, and frequency domain transformations do not apply.

While the analysis of nonlinear control systems is difficult, serious efforts have been made to develop appropriate theoretical tools for it. Many methods of nonlinear control system analysis have been proposed. Let us briefly describe some of these methods before discussing their details in the following chapters.

Phase plane analysis

Phase plane analysis, discussed in chapter 2, is a graphical method of studying second-order nonlinear systems. Its basic idea is to solve a second order differential equation graphically, instead of seeking an analytical solution. The result is a family of system motion trajectories on a two-dimensional plane, called the phase plane, which allow us to visually observe the motion patterns of the system. While phase plane analysis has a number of important advantages, it has the fundamental disadvantage of being applicable only to systems which can be well approximated by a second-order dynamics. Because of its graphical nature, it is frequently used to provide intuitive insights about nonlinear effects.

Lyapunov theory

Basic Lyapunov theory comprises two methods introduced by Lyapunov, the indirect method and the direct method. The indirect method, or linearization method, states that the stability properties of a nonlinear system in the close vicinity of an equilibrium point are essentially the same as those of its linearized approximation. The method serves as the theoretical justification for using linear control for physical systems, which are always inherently nonlinear. The direct method is a powerful tool for nonlinear system analysis, and therefore the so-called Lyapunov analysis often actually refers to the direct method. The direct method is a generalization of the energy concepts associated with a mechanical system: the motion of a mechanical system is stable if its total mechanical energy decreases all the time. In using the direct method to analyze the stability of a nonlinear system, the idea is to construct a scalar energy-like function (a Lyapunov function) for the system, and to see whether it decreases. The power of this method comes from its generality: it is applicable to all kinds of control systems, be they time-varying or time-invariant, finite dimensional or infinite dimensional. Conversely, the limitation of the method lies in the fact that it is often difficult to find a Lyapunov function for a given system.

Although Lyapunov's direct method is originally a method of stability analysis, it can be used for other problems in nonlinear control. One important application is the design of nonlinear controllers. The idea is to somehow formulate a scalar positive function of the system states, and then choose a control law to make this function decrease. A nonlinear control system thus designed will be guaranteed to be stable. Such a design approach has been used to solve many complex design problems, *e.g.*, in robotics and adaptive control. The direct method can also be used to estimate the performance of a control system and study its robustness. The important subject of Lyapunov analysis is studied in chapters 3 and 4, with chapter 3 presenting the main concepts and results in Lyapunov theory, and chapter 4 discussing some advanced topics.

Describing functions

The describing function method is an approximate technique for studying nonlinear systems. The basic idea of the method is to approximate the nonlinear components in nonlinear control systems by linear "equivalents", and then use frequency domain techniques to analyze the resulting systems. Unlike the phase plane method, it is not restricted to second-order systems. Unlike Lyapunov methods, whose applicability to a specific system hinges on the success of a trial-and-error search for a Lyapunov function, its application is straightforward for nonlinear systems satisfying some easy-to-check conditions.

The method is mainly used to predict limit cycles in nonlinear systems. Other applications include the prediction of subharmonic generation and the determination of system response to sinusoidal excitation. The method has a number of advantages. First, it can deal with low order and high order systems with the same straightforward procedure. Second, because of its similarity to frequency-domain analysis of linear systems, it is conceptually simple and physically appealing, allowing users to exercise their physical and engineering insights about the control system. Third, it can deal with the "hard nonlinearities" frequently found in control systems without any difficulty. As a result, it is an important tool for practical problems of nonlinear control analysis and design. The disadvantages of the method are linked to its approximate nature, and include the possibility of inaccurate predictions (false predictions may be made if certain conditions are not satisfied) and restrictions on the systems to which it applies (for example, it has difficulties in dealing with systems with multiple nonlinearities).

Chapter 2
Phase Plane Analysis

Phase plane analysis is a graphical method for studying second-order systems, which was introduced well before the turn of the century by mathematicians such as Henri Poincare. The basic idea of the method is to generate, in the state space of a second-order dynamic system (a two-dimensional plane called the phase plane), motion trajectories corresponding to various initial conditions, and then to examine the qualitative features of the trajectories. In such a way, information concerning stability and other motion patterns of the system can be obtained. In this chapter, our objective is to gain familiarity with nonlinear systems through this simple graphical method.

Phase plane analysis has a number of useful properties. First, as a graphical method, it allows us to visualize what goes on in a nonlinear system starting from various initial conditions, without having to solve the nonlinear equations analytically. Second, it is not restricted to small or smooth nonlinearities, but applies equally well to strong nonlinearities and to "hard" nonlinearities. Finally, some practical control systems can indeed be adequately approximated as second-order systems, and the phase plane method can be used easily for their analysis. Conversely, of course, the fundamental disadvantage of the method is that it is restricted to second-order (or first-order) systems, because the graphical study of higher-order systems is computationally and geometrically complex.

2.1 Concepts of Phase Plane Analysis

2.1.1 Phase Portraits

The phase plane method is concerned with the graphical study of second-order autonomous systems described by

$$\dot{x}_1 = f_1(x_1, x_2) \tag{2.1a}$$

$$\dot{x}_2 = f_2(x_1, x_2) \tag{2.1b}$$

where x_1 and x_2 are the states of the system, and f_1 and f_2 are nonlinear functions of the states. Geometrically, the state space of this system is a plane having x_1 and x_2 as coordinates. We will call this plane the *phase plane*.

Given a set of initial conditions $\mathbf{x}(0) = \mathbf{x}_o$, Equation (2.1) defines a solution $\mathbf{x}(t)$. With time t varied from zero to infinity, the solution $\mathbf{x}(t)$ can be represented geometrically as a curve in the phase plane. Such a curve is called a phase plane *trajectory*. A family of phase plane trajectories corresponding to various initial conditions is called a *phase portrait* of a system.

To illustrate the concept of phase portrait, let us consider the following simple system.

Example 2.1: Phase portrait of a mass-spring system

The governing equation of the mass-spring system in Figure 2.1(a) is the familiar linear second-order differential equation

$$\ddot{x} + x = 0 \tag{2.2}$$

Assume that the mass is initially at rest, at length x_o. Then the solution of the equation is

$$x(t) = x_o \cos t$$

$$\dot{x}(t) = -x_o \sin t$$

Eliminating time t from the above equations, we obtain the equation of the trajectories

$$x^2 + \dot{x}^2 = x_o^2$$

This represents a circle in the phase plane. Corresponding to different initial conditions, circles of different radii can be obtained. Plotting these circles on the phase plane, we obtain a phase portrait for the mass-spring system (Figure 2.1.b). □

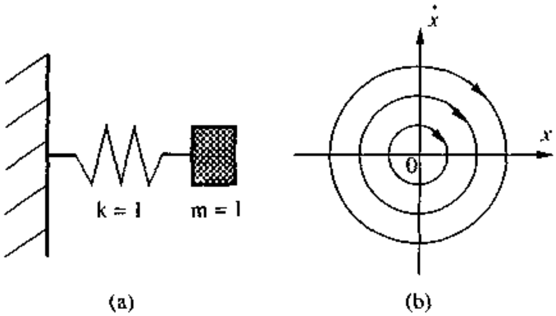


Figure 2.1 : A mass-spring system and its phase portrait

The power of the phase portrait lies in the fact that once the phase portrait of a system is obtained, the nature of the system response corresponding to various initial conditions is directly displayed on the phase plane. In the above example, we easily see that the system trajectories neither converge to the origin nor diverge to infinity. They simply circle around the origin, indicating the marginal nature of the system's stability.

A major class of second-order systems can be described by differential equations of the form

$$\ddot{x} + f(x, \dot{x}) = 0 \tag{2.3}$$

In state space form, this dynamics can be represented as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -f(x_1, x_2)$$

with $x_1 = x$ and $x_2 = \dot{x}$. Most second-order systems in practice, such as mass-damper-spring systems in mechanics, or resistor-coil-capacitor systems in electrical engineering, can be represented in or transformed into this form. For these systems, the states are x and its derivative $\dot{x}$. Traditionally, the phase plane method is developed for the dynamics (2.3), and the phase plane is defined as the plane having x and $\dot{x}$ as coordinates. But it causes no difficulty to extend the method to more general dynamics of the form (2.1), with the (x_1 , x_2) plane as the phase plane, as we do in this chapter.

2.1.2 Singular Points

An important concept in phase plane analysis is that of a singular point. A singular point is an equilibrium point in the phase plane. Since an equilibrium point is defined as a point where the system states can stay forever, this implies that $\dot{\mathbf{x}} = \mathbf{0}$, and using (2.1),

$$f_1(x_1, x_2) = 0 \qquad f_2(x_1, x_2) = 0 \tag{2.4}$$

The values of the equilibrium states can be solved from (2.4).

For a linear system, there is usually only one singular point (although in some cases there can be a *continuous* set of singular points, as in the system $\ddot{x} + \dot{x} = 0$, for which all points on the real axis are singular points). However, a nonlinear system often has more than one isolated singular point, as the following example shows.

Example 2.2: A nonlinear second-order system

Consider the system

$$\ddot{x} + 0.6\,\dot{x} + 3\,x + x^2 = 0$$

whose phase portrait is plotted in Figure 2.2. The system has two singular points, one at (0, 0) and the other at (−3, 0). The motion patterns of the system trajectories in the vicinity of the two singular points have different natures. The trajectories move towards the point $x = 0$ while moving away from the point $x = -3$. □

One may wonder why an equilibrium point of a second-order system is called a *singular* point. To answer this, let us examine the slope of the phase trajectories. From (2.1), the slope of the phase trajectory passing through a point (x_1, x_2) is determined by

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \tag{2.5}$$

With the functions f_1 and f_2 assumed to be single valued, there is usually a definite value for this slope at any given point in phase plane. This implies that the phase trajectories will not intersect. At singular points, however, the value of the slope is 0/0, *i.e.*, the slope is indeterminate. Many trajectories may intersect at such points, as seen from Figure 2.2. This indeterminacy of the slope accounts for the adjective "singular".

Singular points are very important features in the phase plane. Examination of the singular points can reveal a great deal of information about the properties of a

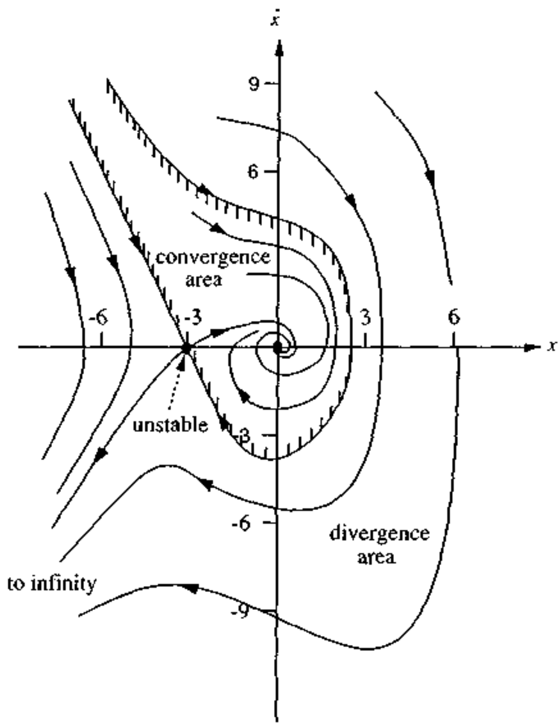


Figure 2.2 : The phase portrait of a nonlinear system

system. In fact, the stability of linear systems is uniquely characterized by the nature of their singular points. For nonlinear systems, besides singular points, there may be more complex features, such as limit cycles. These issues will be discussed in detail in sections 2.3 and 2.4.

Note that, although the phase plane method is developed primarily for second-order systems, it can also be applied to the analysis of first-order systems of the form

$$\dot{x} + f(x) = 0$$

The idea is still to plot $\dot{x}$ with respect to x in the phase plane. The difference now is that the phase portrait is composed of a single trajectory.

Example 2.3: A first-order system

Consider the system

$$\dot{x} = -4x + x^3$$

There are three singular points, defined by $-4x + x^3 = 0$, namely, $x = 0, -2,$ and 2. The phase-portrait of the system consists of a single trajectory, and is shown in Figure 2.3. The arrows in the figure denote the direction of motion, and whether they point toward the left or the right at a particular point is determined by the sign of $\dot{x}$ at that point. It is seen from the phase portrait of this system that the equilibrium point $x = 0$ is stable, while the other two are unstable. □

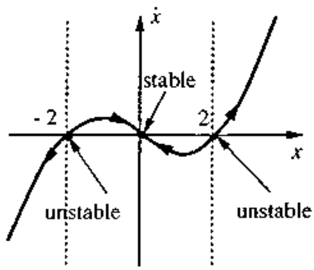


Figure 2.3 : Phase trajectory of a first-order system

2.1.3 Symmetry in Phase Plane Portraits

A phase portrait may have *a priori* known symmetry properties, which can simplify its generation and study. If a phase portrait is symmetric with respect to the x_1 or the x_2 axis, one only needs in practice to study half of it. If a phase portrait is symmetric with respect to both the x_1 and x_2 axes, only one quarter of it has to be explicitly considered.

Before generating a phase portrait itself, we can determine its symmetry properties by examining the system equations. Let us consider the second-order dynamics (2.3). The slope of trajectories in the phase plane is of the form

$$\frac{dx_2}{dx_1} = -\frac{f(x_1, x_2)}{\dot{x}}$$

Since symmetry of the phase portraits also implies symmetry of the slopes (equal in absolute value but opposite in sign), we can identify the following situations:

Symmetry about the x_1 axis: The condition is

$$f(x_1, x_2) = f(x_1, -x_2)$$

This implies that the function f should be even in x_2. The mass-spring system in Example 2.1 satisfies this condition. Its phase portrait is seen to be symmetric about the x_1 axis.

Symmetry about the x_2 axis: Similarly,

$$f(x_1, x_2) = -f(-x_1, x_2)$$

implies symmetry with respect to the x_2 axis. The mass-spring system also satisfies this condition.

Symmetry about the origin: When

$$f(x_1, x_2) = -f(-x_1, -x_2)$$

the phase portrait of the system is symmetric about the origin.

2.2 Constructing Phase Portraits

Today, phase portraits are routinely computer-generated. In fact, it is largely the advent of the computer in the early 1960's, and the associated ease of quickly generating phase portraits, which spurred many advances in the study of complex nonlinear dynamic behaviors such as chaos. However, of course (as *e.g.*, in the case of root locus for linear systems), it is still practically useful to learn how to roughly sketch phase portraits or quickly verify the plausibility of computer outputs.

There are a number of methods for constructing phase plane trajectories for linear or nonlinear systems, such as the so-called analytical method, the method of isoclines, the delta method, Lienard's method, and Pell's method. We shall discuss two of them in this section, namely, the analytical method and the method of isoclines. These methods are chosen primarily because of their relative simplicity. The analytical method involves the analytical solution of the differential equations describing the systems. It is useful for some special nonlinear systems, particularly piece-wise linear systems, whose phase portraits can be constructed by piecing together the phase portraits of the related linear systems. The method of isoclines is a graphical method which can conveniently be applied to construct phase portraits for systems which cannot be solved analytically, which represent by far the most common case.

ANALYTICAL METHOD

There are two techniques for generating phase plane portraits analytically. Both techniques lead to a functional relation between the two phase variables x_1 and x_2 in the form

$$g(x_1, x_2, c) = 0 \tag{2.6}$$

where the constant c represents the effects of initial conditions (and, possibly, of external input signals). Plotting this relation in the phase plane for different initial conditions yields a phase portrait.

The first technique involves solving equations (2.1) for x_1 and x_2 as functions of time t, *i.e.*,

$$x_1(t) = g_1(t) \qquad x_2(t) = g_2(t)$$

and then eliminating time t from these equations, leading to a functional relation in the form of (2.6). This technique was already illustrated in Example 2.1.

The second technique, on the other hand, involves directly eliminating the time variable, by noting that

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

and then solving this equation for a functional relation between x_1 and x_2. Let us use this technique to solve the mass-spring equation again.

Example 2.4: Mass-spring system

By noting that $\ddot{x} = (d\dot{x}/dx)(dx/dt)$, we can rewrite (2.2) as

$$\dot{x}\frac{d\dot{x}}{dx} + x = 0$$

Integration of this equation yields

$$\dot{x}^2 + x^2 = x_o^2$$ □

One sees that the second technique is more straightforward in generating the equations for the phase plane trajectories.

Most nonlinear systems cannot be easily solved by either of the above two techniques. However, for piece-wise linear systems, an important class of nonlinear systems, this method can be conveniently used, as the following example shows.

Example 2.5: A satellite control system

Figure 2.4 shows the control system for a simple satellite model. The satellite, depicted in Figure 2.5(a), is simply a rotational unit inertia controlled by a pair of thrusters, which can provide either a positive constant torque U (positive firing) or a negative torque $-U$ (negative firing). The purpose of the control system is to maintain the satellite antenna at a zero angle by appropriately firing the thrusters. The mathematical model of the satellite is

$$\ddot{\theta} = u$$

where u is the torque provided by the thrusters and θ is the satellite angle.

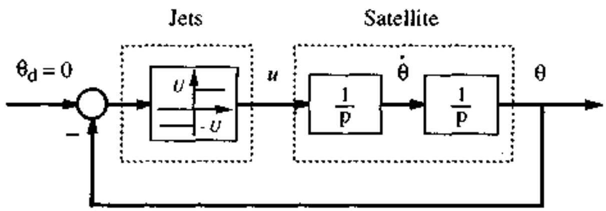


Figure 2.4 : Satellite control system

Let us examine on the phase plane the behavior of the control system when the thrusters are fired according to the control law

$$u(t) = \begin{cases} -U & \text{if } \theta > 0 \\ U & \text{if } \theta < 0 \end{cases} \tag{2.7}$$

which means that the thrusters push in the counterclockwise direction if θ is positive, and vice versa.

As the first step of the phase portrait generation, let us consider the phase portrait when the thrusters provide a positive torque U. The dynamics of the system is

$$\ddot{\theta} = U$$

which implies that $\dot{\theta}\, d\dot{\theta} = U\, d\theta$. Therefore, the phase trajectories are a family of parabolas defined by

$$\dot{\theta}^2 = 2U\theta + c_1$$

where c_1 is a constant. The corresponding phase portrait of the system is shown in Figure 2.5(b).

When the thrusters provide a negative torque $-U$, the phase trajectories are similarly found to be

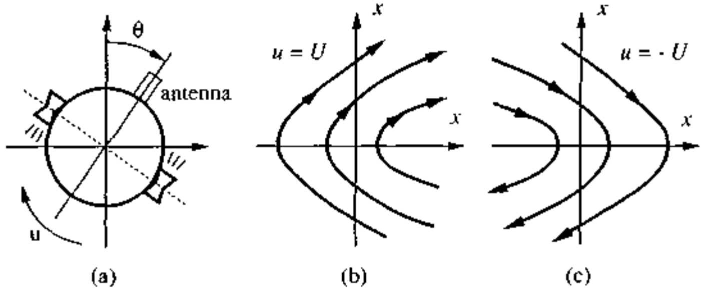


Figure 2.5 : Satellite control using on-off thrusters

$$\dot{\theta}^2 = -2Ux + c_1$$

with the corresponding phase portrait shown in Figure 2.5(c).

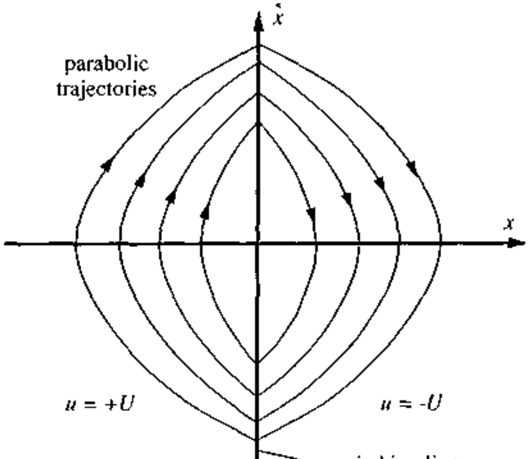


Figure 2.6 : Complete phase portrait of the control system

The complete phase portrait of the closed-loop control system can be obtained simply by connecting the trajectories on the left half of the phase plane in 2.5(b) with those on the right half of the phase plane in 2.5(c), as shown in Figure 2.6. The vertical axis represents a switching line, because the control input and thus the phase trajectories are switched on that line. It is interesting to see that, starting from a nonzero initial angle, the satellite will oscillate in periodic motions under the action of the jets. One concludes from this phase portrait that the system is marginally stable, similarly to the mass-spring system in Example 2.1. Convergence of the system to the zero angle can be obtained by adding rate feedback (Exercise 2.4). □

THE METHOD OF ISOCLINES

The basic idea in this method is that of isoclines. Consider the dynamics in (2.1). At a point (x_1, x_2) in the phase plane, the slope of the tangent to the trajectory can be determined by (2.5). An isocline is defined to be the locus of the points with a given tangent slope. An isocline with slope α is thus defined to be

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = \alpha$$

This is to say that points on the curve

$$f_2(x_1, x_2) = \alpha f_1(x_1, x_2)$$

all have the same tangent slope α.

In the method of isoclines, the phase portrait of a system is generated in two steps. In the first step, a field of directions of tangents to the trajectories is obtained. In the second step, phase plane trajectories are formed from the field of directions .

Let us explain the isocline method on the mass-spring system in (2.2). The slope of the trajectories is easily seen to be

$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2}$$

Therefore, the isocline equation for a slope α is

$$x_1 + \alpha\, x_2 = 0$$

i.e., a straight line. Along the line, we can draw a lot of short line segments with slope α. By taking α to be different values, a set of isoclines can be drawn, and a field of directions of tangents to trajectories are generated, as shown in Figure 2.7. To obtain trajectories from the field of directions, we assume that the the tangent slopes are locally constant. Therefore, a trajectory starting from any point in the plane can be found by connecting a sequence of line segments.

Let us use the method of isoclines to study the Van der Pol equation, a nonlinear equation.

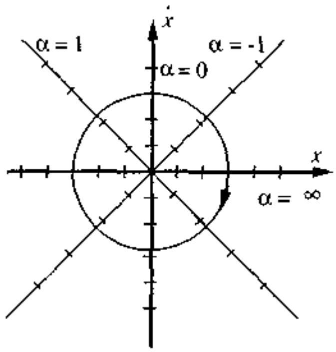


Figure 2.7 : Isoclines for the mass-spring system

Example 2.6: The Van der Pol equation

For the Van der Pol equation

$$\ddot{x} + 0.2\,(x^2 - 1)\dot{x} + x = 0$$

an isocline of slope α is defined by

$$\frac{d\dot{x}}{dx} = -\frac{0.2\,(x^2 - 1)\,\dot{x} + x}{\dot{x}} = \alpha$$

Therefore, the points on the curve

$$0.2\,(x^2 - 1\,)\dot{x} + x + \alpha\dot{x} = 0$$

all have the same slope α.

By taking α of different values, different isoclines can be obtained, as plotted in Figure 2.8. Short line segments are drawn on the isoclines to generate a field of tangent directions. The phase portraits can then be obtained, as shown in the plot. It is interesting to note that there exists a closed curve in the portrait, and the trajectories starting from both outside and inside converge to this curve. This closed curve corresponds to a limit cycle, as will be discussed further in section 2.5. □

Note that the same scales should be used for the x_1 axis and x_2 axis of the phase plane, so that the derivative dx_2/dx_1 equals the geometric slope of the trajectories. Also note that, since in the second step of phase portrait construction we essentially assume that the slope of the phase plane trajectories is locally constant, more isoclines should be plotted in regions where the slope varies quickly, to improve accuracy.

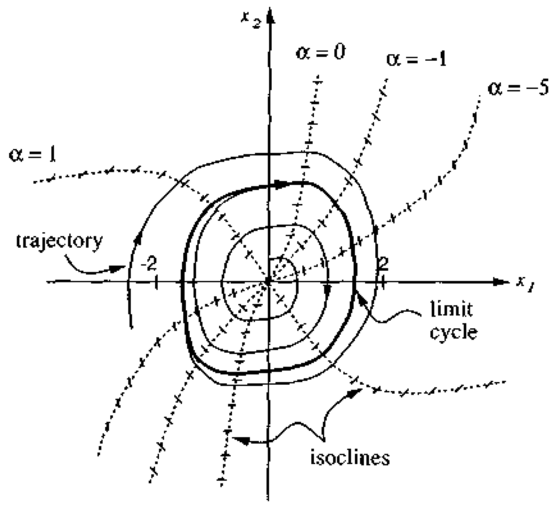


Figure 2.8 : Phase portrait of the Van der Pol equation

2.3 Determining Time from Phase Portraits

Note that time t does not explicitly appear in the phase plane having x_1 and x_2 as coordinates. However, in some cases, we might be interested in the time information. For example, one might want to know the time history of the system states starting from a specific initial point. Another relevant situation is when one wants to know how long it takes for the system to move from a point to another point in a phase plane trajectory. We now describe two techniques for computing time history from phase portraits. Both techniques involve a step-by step procedure for recovering time.

Obtaining time from $\Delta t \approx \Delta x / \dot{x}$

In a short time Δt, the change of x is approximately

$$\Delta x \approx \dot{x} \Delta t \tag{2.8}$$

where $\dot{x}$ is the velocity corresponding to the increment Δx. Note that for a Δx of finite magnitude, the average value of velocity during a time increment should be used to improve accuracy. From (2.8), the length of time corresponding to the increment Δx

is

$$\Delta t \approx \frac{\Delta x}{\dot{x}}$$

The above reasoning implies that, in order to obtain the time corresponding to the motion from one point to another point along a trajectory, one should divide the corresponding part of the trajectory into a number of small segments (not necessarily equally spaced), find the time associated with each segment, and then add up the results. To obtain the time history of states corresponding to a certain initial condition, one simply computes the time t for each point on the phase trajectory, and then plots x with respect to t and $\dot{x}$ with respect to t.

Obtaining time from $t = \int (1/\dot{x})\, dx$

Since $\dot{x} = dx/dt$, we can write $dt = dx/\dot{x}$. Therefore,

$$t - t_o = \int_{x_o}^{x} (1/\dot{x})\, dx$$

where x corresponds to time t and x_o corresponds to time t_o. This equation implies that, if we plot a phase plane portrait with new coordinates x and $(1/\dot{x})$, then the area under the resulting curve is the corresponding time interval.

2.4 Phase Plane Analysis of Linear Systems

In this section, we describe the phase plane analysis of linear systems. Besides allowing us to visually observe the motion patterns of linear systems, this will also help the development of nonlinear system analysis in the next section, because a nonlinear systems behaves similarly to a linear system around each equilibrium point.

The general form of a linear second-order system is

$$\dot{x}_1 = a x_1 + b x_2 \tag{2.9a}$$

$$\dot{x}_2 = c x_1 + d x_2 \tag{2.9b}$$

To facilitate later discussions, let us transform this equation into a scalar second-order differential equation. Note from (2.9a) and (2.9b) that

$$b\,\dot{x}_2 = b\,c x_1 + d(\dot{x}_1 - a x_1)$$

Consequently, differentiation of (2.9a) and then substitution of (2.9b) leads to

$$\ddot{x}_1 = (a+d)\dot{x}_1 + (cb - ad)x_1$$

Therefore, we will simply consider the second-order linear system described by

$$\ddot{x} + a\dot{x} + bx = 0 \tag{2.10}$$

To obtain the phase portrait of this linear system, we first solve for the time history

$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \quad \text{for } \lambda_1 \neq \lambda_2 \tag{2.11a}$$

$$x(t) = k_1 e^{\lambda_1 t} + k_2 t\, e^{\lambda_1 t} \quad \text{for } \lambda_1 = \lambda_2 \tag{2.11b}$$

where the constants λ_1 and λ_2 are the solutions of the characteristic equation

$$s^2 + as + b = (s - \lambda_1)\,(s - \lambda_2) = 0$$

The roots λ_1 and λ_2 can be explicitly represented as

$$\lambda_1 = (-a + \sqrt{a^2 - 4b}\,)/2 \qquad \lambda_2 = (-a - \sqrt{a^2 - 4b}\,)/2$$

For linear systems described by (2.10), there is only one singular point (assuming $b \neq 0$), namely the origin. However, the trajectories in the vicinity of this singularity point can display quite different characteristics, depending on the values of a and b. The following cases can occur

1. λ_1 and λ_2 are both real and have the same sign (positive or negative)
2. λ_1 and λ_2 are both real and have opposite signs
3. λ_1 and λ_2 are complex conjugate with non-zero real parts
4. λ_1 and λ_2 are complex conjugates with real parts equal to zero

We now briefly discuss each of the above four cases.

STABLE OR UNSTABLE NODE

The first case corresponds to a *node*. A node can be stable or unstable. If the eigenvalues are negative, the singularity point is called a *stable node* because both $x(t)$ and $\dot{x}(t)$ converge to zero exponentially, as shown in Figure 2.9(a). If both eigenvalues are positive, the point is called an *unstable node*, because both $x(t)$ and $\dot{x}(t)$ diverge from zero exponentially, as shown in Figure 2.9(b). Since the eigenvalues are real, there is no oscillation in the trajectories.

SADDLE POINT

The second case (say $\lambda_1 < 0$ and $\lambda_2 > 0$) corresponds to a *saddle point* (Figure 2.9(c)). The phase portrait of the system has the interesting "saddle" shape shown in Figure 2.9(c). Because of the unstable pole λ_2, almost all of the system trajectories diverge to infinity. In this figure, one also observes two straight lines passing through the origin. The diverging line (with arrows pointing to infinity) corresponds to initial conditions which make k_2 (*i.e.*, the unstable component) equal zero. The converging straight line corresponds to initial conditions which make k_1 equal zero.

STABLE OR UNSTABLE FOCUS

The third case corresponds to a focus. A *stable focus* occurs when the real part of the eigenvalues is negative, which implies that $x(t)$ and $\dot{x}(t)$ both converge to zero. The system trajectories in the vicinity of a stable focus are depicted in Figure 2.9(d). Note that the trajectories encircle the origin one or more times before converging to it, unlike the situation for a stable node. If the real part of the eigenvalues is positive, then $x(t)$ and $\dot{x}(t)$ both diverge to infinity, and the singularity point is called an *unstable focus*. The trajectories corresponding to an unstable focus are sketched in Figure 2.9(e).

CENTER POINT

The last case corresponds to a center point, as shown in Figure 2.9(f). The name comes from the fact that all trajectories are ellipses and the singularity point is the center of these ellipses. The phase portrait of the undamped mass-spring system belongs to this category.

Note that the stability characteristics of linear systems are uniquely determined by the nature of their singularity points. This, however, is not true for nonlinear systems.

2.5 Phase Plane Analysis of Nonlinear Systems

In discussing the phase plane analysis of nonlinear systems, two points should be kept in mind. Phase plane analysis of nonlinear systems is related to that of linear systems, because the local behavior of a nonlinear system can be approximated by the behavior of a linear system. Yet, nonlinear systems can display much more complicated patterns in the phase plane, such as multiple equilibrium points and limit cycles. We now discuss these points in more detail.

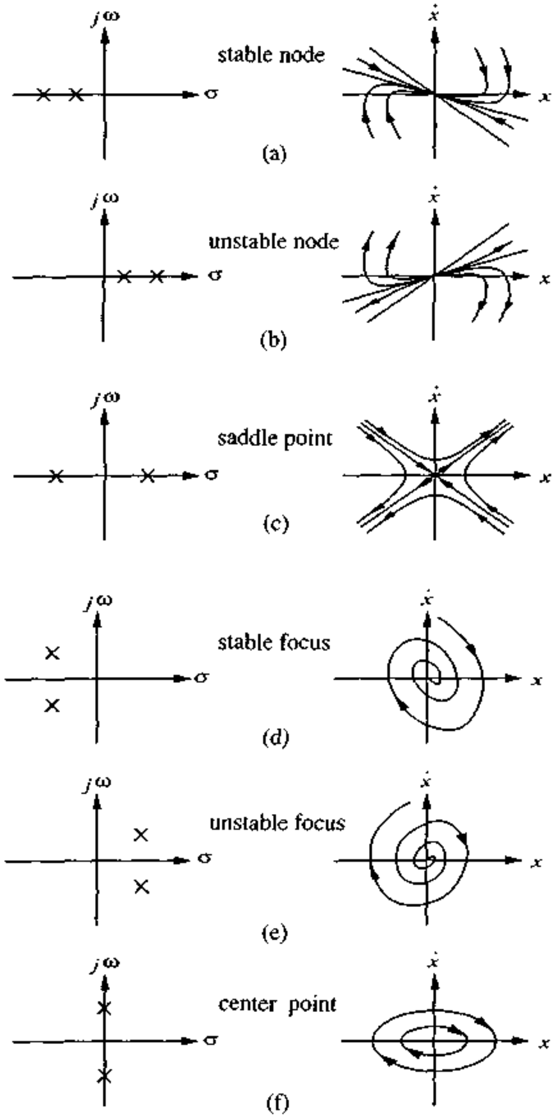


Figure 2.9 : Phase-portraits of linear systems

LOCAL BEHAVIOR OF NONLINEAR SYSTEMS

In the phase portrait of Figure 2.2, one notes that, in contrast to linear systems, there are two singular points, $(0, 0)$ and $(-3, 0)$. However, we also note that the features of the phase trajectories in the neighborhood of the two singular points look very much like those of linear systems, with the first point corresponding to a stable focus and the second to a saddle point. This similarity to a linear system in the local region of each singular point can be formalized by linearizing the nonlinear system, as we now discuss.

If the singular point of interest is not at the origin, by defining the difference between the original state and the singular point as a new set of state variables, one can always shift the singular point to the origin. Therefore, without loss of generality, we may simply consider Equation (2.1) with a singular point at 0. Using Taylor expansion, Equations (2.1a) and (2.1b) can be rewritten as

$$\dot{x}_1 = a x_1 + b x_2 + g_1(x_1, x_2)$$

$$\dot{x}_2 = c x_1 + d x_2 + g_2(x_1, x_2)$$

where g_1 and g_2 contain higher order terms.

In the vicinity of the origin, the higher order terms can be neglected, and therefore, the nonlinear system trajectories essentially satisfy the linearized equation

$$\dot{x}_1 = a\, x_1 + b\, x_2$$

$$\dot{x}_2 = c\, x_1 + d\, x_2$$

As a result, the local behavior of the nonlinear system can be approximated by the patterns shown in Figure 2.9.

LIMIT CYCLES

In the phase portrait of the nonlinear Van der Pol equation, shown in Figure 2.8, one observes that the system has an unstable node at the origin. Furthermore, there is a closed curve in the phase portrait. Trajectories inside the curve and those outside the curve all tend to this curve, while a motion started on this curve will stay on it forever, circling periodically around the origin. This curve is an instance of the so-called "limit cycle" phenomenon. Limit cycles are unique features of nonlinear systems.

In the phase plane, *a limit cycle is defined as an isolated closed curve.* The trajectory has to be both closed, indicating the periodic nature of the motion, and isolated, indicating the limiting nature of the cycle (with nearby trajectories converging or diverging from it). Thus, while there are many closed curves in the phase portraits of the mass-spring-damper system in Example 2.1 or the satellite system in Example 2.5, these are not considered limit cycles in this definition, because they are not isolated.

Depending on the motion patterns of the trajectories in the vicinity of the limit cycle, one can distinguish three kinds of limit cycles

1. **Stable Limit Cycles**: all trajectories in the vicinity of the limit cycle converge to it as $t \to \infty$ (Figure 2.10(a));
2. **Unstable Limit Cycles**: all trajectories in the vicinity of the limit cycle diverge from it as $t \to \infty$ (Figure 2.10(b));
3. **Semi-Stable Limit Cycles**: some of the trajectories in the vicinity converge to it, while the others diverge from it as $t \to \infty$ (Figure 2.10(c));

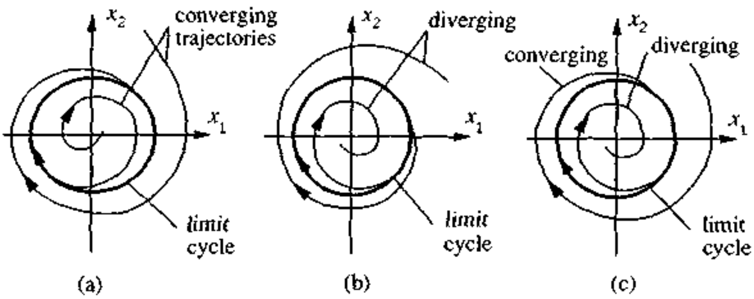


Figure 2.10 : Stable, unstable, and semi-stable limit cycles

As seen from the phase portrait of Figure 2.8, the limit cycle of the Van der Pol equation is clearly stable. Let us consider some additional examples of stable, unstable, and semi-stable limit cycles.

Example 2.7: stable, unstable, and semi-stable limit cycles

Consider the following nonlinear systems

(a) $\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1) \qquad \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)$ (2.12)

(b) $\dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2 - 1) \qquad \dot{x}_2 = -x_1 + x_2(x_1^2 + x_2^2 - 1)$ (2.13)

(c) $\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)^2 \qquad \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)^2$ (2.14)

Let us study system (a) first. By introducing polar coordinates

$$r = (x_1^2 + x_2^2)^{1/2} \qquad \theta = \tan^{-1}(x_2/x_1)$$

the dynamic equations (2.12) are transformed as

$$\frac{dr}{dt} = -r(r^2 - 1) \qquad \frac{d\theta}{dt} = -1$$

When the state starts on the unit circle, the above equation shows that $\dot{r}(t) = 0$. Therefore, the state will circle around the origin with a period $1/2\pi$. When $r < 1$, then $\dot{r} > 0$. This implies that the state tends to the circle from inside. When $r > 1$, then $\dot{r} < 0$. This implies that the state tends toward the unit circle from outside. Therefore, the unit circle is a stable limit cycle. This can also be concluded by examining the analytical solution of (2.12)

$$r(t) = \frac{1}{(1 + c_o e^{-2t})^{1/2}} \qquad \theta(t) = \theta_o - t$$

where

$$c_o = \frac{1}{r_o^2} - 1$$

Similarly, one can find that the system (b) has an unstable limit cycle and system (c) has a semi-stable limit cycle. □

2.6 Existence of Limit Cycles

As mentioned in chapter 1, it is of great importance for control engineers to predict the existence of limit cycles in control systems. In this section, we state three simple classical theorems to that effect. These theorems are easy to understand and apply.

The first theorem to be presented reveals a simple relationship between the existence of a limit cycle and the number of singular points it encloses. In the statement of the theorem, we use N to represent the number of nodes, centers, and foci enclosed by a limit cycle, and S to represent the number of enclosed saddle points.

Theorem 2.1 (Poincare) *If a limit cycle exists in the second-order autonomous system (2.1), then* $N = S + 1$.

This theorem is sometimes called the *index theorem*. Its proof is mathematically involved (actually, a family of such proofs led to the development of algebraic topology) and shall be omitted here. One simple inference from this theorem is that a limit cycle must enclose at least one equilibrium point. The theorem's result can be verified easily on Figures 2.8 and 2.10.

The second theorem is concerned with the asymptotic properties of the trajectories of second-order systems.

Theorem 2.2 (Poincare-Bendixson) *If a trajectory of the second-order autonomous system remains in a finite region* Ω, *then one of the following is true:*

(a) the trajectory goes to an equilibrium point

(b) the trajectory tends to an asymptotically stable limit cycle

(c) the trajectory is itself a limit cycle

While the proof of this theorem is also omitted here, its intuitive basis is easy to see, and can be verified on the previous phase portraits.

The third theorem provides a sufficient condition for the non-existence of limit cycles.

Theorem 2.3 (Bendixson) *For the nonlinear system (2.1), no limit cycle can exist in a region* Ω *of the phase plane in which* $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ *does not vanish and does not change sign.*

Proof: Let us prove this theorem by contradiction. First note that, from (2.5), the equation

$$f_2\, dx_1 - f_1\, dx_2 = 0 \tag{2.15}$$

is satisfied for any system trajectories, including a limit cycle. Thus, along the closed curve L of a limit cycle, we have

$$\int_L (f_1\, dx_2 - f_2\, dx_1) = 0 \tag{2.16}$$

Using Stokes' Theorem in calculus, we have

$$\int_L (f_1\, dx_2 - f_2\, dx_1) = \int\int \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1\, dx_2 = 0$$

where the integration on the right-hand side is carried out on the area enclosed by the limit cycle.

By Equation (2.16), the left-hand side must equal zero. This, however, contradicts the fact that the right-hand side cannot equal zero because by hypothesis $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ does not vanish and does not change sign. □

Let us illustrate the result on an example.

Example 2.8: Consider the nonlinear system

$$\dot{x}_1 = g(x_2) + 4x_1 x_2^2$$

$$\dot{x}_2 = h(x_1) + 4x_1^2 x_2$$

Since

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 4(x_1^2 + x_2^2)$$

which is always strictly positive (except at the origin), the system does not have any limit cycles anywhere in the phase plane. □

The above three theorems represent very powerful results. It is important to notice, however, that they have no equivalent in higher-order systems, where exotic asymptotic behaviors other than equilibrium points and limit cycles can occur.

2.7 Summary

Phase plane analysis is a graphical method used to study second-order dynamic systems. The major advantage of the method is that it allows visual examination of the global behavior of systems. The major disadvantage is that it is mainly limited to second-order systems (although extensions to third-order systems are often achieved with the aid of computer graphics). The phenomena of multiple equilibrium points and of limit cycles are clearly seen in phase plane analysis. A number of useful classical theorems for the prediction of limit cycles in second-order systems are also presented.

2.8 Notes and References

Phase plane analysis is a very classical topic which has been addressed by numerous control texts. An extensive treatment can be found in [Graham and McRuer, 1961]. Examples 2.2 and 2.3 are adapted from [Ogata, 1970]. Examples 2.5 and 2.6 and section 2.6 are based on [Hsu and Meyer, 1968].

2.9 Exercises

2.1 Draw the phase portrait and discuss the properties of the linear, unity feedback control system of open-loop transfer function

$$G(p) = \frac{10}{p\,(1 + 0.1\,p)}$$

2.2 Draw the phase portraits of the following systems, using isoclines

(*a*) $\ddot{\theta} + \dot{\theta} + 0.5\,\theta = 0$

(*b*) $\ddot{\theta} + \dot{\theta} + 0.5\,\theta = 1$

(*c*) $\ddot{\theta} + \dot{\theta}^2 + 0.5\,\theta = 0$

2.3 Consider the nonlinear system

$$\dot{x} = y + x\,(x^2 + y^2 - 1)\sin\frac{1}{x^2 + y^2 - 1}$$

$$\dot{y} = -x + y\,(x^2 + y^2 - 1)\sin\frac{1}{x^2 + y^2 - 1}$$

Without solving the above equations explicitly, show that the system has infinite number of limit cycles. Determine the stability of these limit cycles. (*Hint:* Use polar coordinates.)

2.4 The system shown in Figure 2.10 represents a satellite control system with rate feedback provided by a gyroscope. Draw the phase portrait of the system, and determine the system's stability.

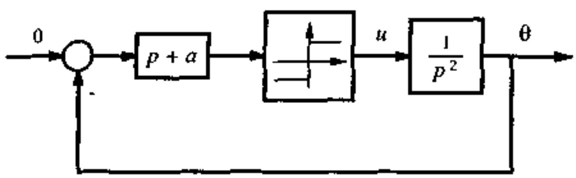


Figure 2.10 : Satellite control system with rate feedback

Chapter 3
Fundamentals of Lyapunov Theory

Given a control system, the first and most important question about its various properties is whether it is stable, because an unstable control system is typically useless and potentially dangerous. Qualitatively, a system is described as stable if starting the system somewhere near its desired operating point implies that it will stay around the point ever after. The motions of a pendulum starting near its two equilibrium points, namely, the vertical up and down positions, are frequently used to illustrate unstable and stable behavior of a dynamic system. For aircraft control systems, a typical stability problem is intuitively related to the following question: will a trajectory perturbation due to a gust cause a significant deviation in the later flight trajectory? Here, the desired operating point of the system is the flight trajectory in the absence of disturbance. Every control system, whether linear or nonlinear, involves a stability problem which should be carefully studied.

The most useful and general approach for studying the stability of nonlinear control systems is the theory introduced in the late 19^{th} century by the Russian mathematician Alexandr Mikhailovich Lyapunov. Lyapunov's work, *The General Problem of Motion Stability*, includes two methods for stability analysis (the so-called linearization method and direct method) and was first published in 1892. The linearization method draws conclusions about a nonlinear system's local stability around an equilibrium point from the stability properties of its linear approximation. The direct method is not restricted to local motion, and determines the stability properties of a nonlinear system by constructing a scalar "energy-like" function for the system and examining the function's time variation. For over half a century, however, Lyapunov's pioneering work on stability received little attention outside Russia, although it was translated into French in 1908 (at the instigation of Poincare), and reprinted by Princeton University Press in 1947. The publication of the work of Lur'e and a book by La Salle and Lefschetz brought Lyapunov's work to the attention of the larger control engineering community in the early 1960's. Many refinements of Lyapunov's methods have since been developed. Today, Lyapunov's linearization method has come to represent the theoretical justification of linear control, while Lyapunov's direct method has become the most important tool for nonlinear system analysis and design. Together, the linearization method and the direct method constitute the so-called Lyapunov stability theory.

The objective of this and the next chapter is to present Lyapunov stability theory and illustrate its use in the analysis and the design of nonlinear systems. To prevent mathematical complexity from obscuring the theoretical concepts, this chapter presents the most basic results of Lyapunov theory in terms of autonomous (*i.e.*, time-invariant) systems, leaving more advanced topics to chapter 4. This chapter is organized as follows. In section 3.1, we provide some background definitions concerning nonlinear systems and equilibrium points. In section 3.2, various concepts of stability are described to characterize different aspects of system behavior. Lyapunov's linearization method is presented in section 3.3. The most useful theorems in the direct method are studied in section 3.4. Section 3.5 is devoted to the question of how to use these theorems to study the stability of particular classes of nonlinear systems. Section 3.6 sketches how the direct method can be used as a powerful way of designing controllers for nonlinear systems.

3.1 Nonlinear Systems and Equilibrium Points

Before addressing the main problems of defining and determining stability in the next sections, let us discuss some relatively simple background issues.

NONLINEAR SYSTEMS

A nonlinear dynamic system can usually be represented by a set of nonlinear differential equations in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{3.1}$$

where $\mathbf{f}$ is a $n \times 1$ nonlinear vector function, and $\mathbf{x}$ is the $n \times 1$ state vector. A particular value of the state vector is also called a point because it corresponds to a point in the state-space. The number of states n is called the *order* of the system. A solution $\mathbf{x}(t)$ of the equations (3.1) usually corresponds to a curve in state space as t varies from zero to infinity, as already seen in phase plane analysis for the case $n = 2$. This curve is generally referred to as a *state trajectory* or a *system trajectory.*

It is important to note that although equation (3.1) does not explicitly contain the control input as a variable, it is directly applicable to feedback control systems. The reason is that equation (3.1) can represent the *closed-loop* dynamics of a feedback control system, with the control input being a function of state $\mathbf{x}$ and time t, and therefore disappearing in the closed-loop dynamics. Specifically, if the plant dynamics is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

and some control law has been selected

$$\mathbf{u} = \mathbf{g}(\mathbf{x}, t)$$

then the closed-loop dynamics is

$$\dot{\mathbf{x}} = \mathbf{f}\,[\mathbf{x}, \mathbf{g}(\mathbf{x}, t), t]$$

which can be rewritten in the form (3.1). Of course, equation (3.1) can also represent dynamic systems where no control signals are involved, such as a freely swinging pendulum.

A special class of nonlinear systems are *linear systems*. The dynamics of linear systems are of the form

$$\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x}$$

where A(t) is an $n \times n$ matrix.

AUTONOMOUS AND NON-AUTONOMOUS SYSTEMS

Linear systems are classified as either time-varying or time-invariant, depending on whether the system matrix $\mathbf{A}$ varies with time or not. In the more general context of nonlinear systems, these adjectives are traditionally replaced by "autonomous" and "non-autonomous".

Definition 3.1 *The nonlinear system (3.1) is said to be* <u>*autonomous*</u> *if* $\mathbf{f}$ *does not depend explicitly on time, i.e.*, if the system's state equation can be written

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{3.2}$$

Otherwise, the system is called <u>*non-autonomous*</u> .

Obviously, linear time-invariant (LTI) systems are autonomous and linear time-varying (LTV) systems are non-autonomous. The second-order systems studied in chapter 2 are all autonomous.

Strictly speaking, all physical systems are non-autonomous, because none of their dynamic characteristics is strictly time-invariant. The concept of an autonomous system is an idealized notion, like the concept of a linear system. In practice, however, system properties often change very slowly, and we can neglect their time variation without causing any practically meaningful error.

It is important to note that for control systems, the above definition is made on the *closed-loop dynamics*. Since a control system is composed of a controller and a plant (including sensor and actuator dynamics), the non-autonomous nature of a control system may be due to a time-variation either in the plant or in the control law. Specifically, a time-invariant plant with dynamics

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

may lead to a non-autonomous closed-loop system if a controller dependent on time t is chosen, *i.e.*, if $\mathbf{u} = \mathbf{g}(\mathbf{x}, t)$. For example, the closed-loop system of the simple plant $\dot{x} = -x + u$ can be nonlinear and non-autonomous by choosing u to be nonlinear and time-varying (*e.g.*, $u = -x^2 \sin t$). In fact, adaptive controllers for linear time-invariant plants usually make the closed-loop control systems nonlinear and non-autonomous.

The fundamental difference between autonomous and non-autonomous systems lies in the fact that the state trajectory of an autonomous system is independent of the initial time, while that of a non-autonomous system generally is not. As we will see in the next chapter, this difference requires us to consider the initial time explicitly in defining stability concepts for non-autonomous systems, and makes the analysis more difficult than that of autonomous systems.

It is well known that the analysis of linear time-invariant systems is much easier than that of linear time-varying systems. The same is true with nonlinear systems. Generally speaking, autonomous systems have relatively simpler properties and their analysis is much easier. For this reason, *in the remainder of this chapter, we will concentrate on the analysis of autonomous systems*, represented by (3.2). Extensions of the concepts and results to non-autonomous systems will be studied in chapter 4.

EQUILIBRIUM POINTS

It is possible for a system trajectory to correspond to only a single point. Such a point is called an equilibrium point. As we shall see later, many stability problems are naturally formulated with respect to equilibrium points.

Definition 3.2 *A state* $\mathbf{x}^*$ *is an* *equilibrium state* *(or* *equilibrium point*) *of the system if once* $\mathbf{x}(t)$ *is equal to* $\mathbf{x}^*$, *it remains equal to* $\mathbf{x}^*$ *for all future time.*

Mathematically, this means that the constant vector $\mathbf{x}^*$ satisfies

$$\mathbf{0} = \mathbf{f}(\mathbf{x}^*) \tag{3.3}$$

Equilibrium points can be found by solving the nonlinear algebraic equations (3.3).

A linear time-invariant system

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} \tag{3.4}$$

has a single equilibrium point (the origin $\mathbf{0}$) if $\mathbf{A}$ is nonsingular. If $\mathbf{A}$ is singular, it has an infinity of equilibrium points, which are contained in the null-space of the matrix $\mathbf{A}$, *i.e.*, the subspace defined by $\mathbf{A}\,\mathbf{x} = \mathbf{0}$. This implies that the equilibrium points are not isolated, as reflected by the example $\ddot{x} + \dot{x} = 0$, for which all points on the x axis of the phase plane are equilibrium points.

A nonlinear system can have several (or infinitely many) isolated equilibrium points, as seen in Example 1.1. The following example involves a familiar physical system.

Example 3.1: The Pendulum

Consider the pendulum of Figure 3.1, whose dynamics is given by the following nonlinear autonomous equation

$$MR^2\,\ddot{\theta} + b\,\dot{\theta} + MgR\sin\theta = 0 \tag{3.5}$$

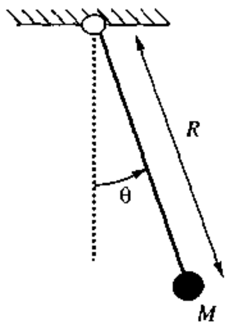


Figure 3.1 : The pendulum

where R is the pendulum's length, M its mass, b the friction coefficient at the hinge, and g the gravity constant. Letting $x_1 = \theta$, $x_2 = \dot{\theta}$, the corresponding state-space equation is

$$\dot{x}_1 = x_2 \tag{3.6a}$$

$$\dot{x}_2 = -\frac{b}{MR^2}x_2 - \frac{g}{R}\sin x_1 \tag{3.6b}$$

Therefore, the equilibrium points are given by

$$x_2 = 0\,, \quad \sin x_1 = 0$$

which leads to the points (0 [2π], 0) and (π [2π], 0). Physically, these points correspond to the pendulum resting exactly at the vertical up and down positions. □

In linear system analysis and design, for notational and analytical simplicity, we often transform the linear system equations in such a way that the equilibrium point is the origin of the state-space. We can do the same thing for nonlinear systems (3.2), about a *specific* equilibrium point. Let us say that the equilibrium point of interest is $\mathbf{x}^*$. Then, by introducing a new variable

$$\mathbf{y} = \mathbf{x} - \mathbf{x}^*$$

and substituting $\mathbf{x} = \mathbf{y} + \mathbf{x}^*$ into equations (3.2), a new set of equations on the variable $\mathbf{y}$ are obtained

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y} + \mathbf{x}^*) \tag{3.7}$$

One can easily verify that there is a one-to-one correspondence between the solutions of (3.2) and those of (3.7), and that in addition, $\mathbf{y} = \mathbf{0}$, the solution corresponding to $\mathbf{x} = \mathbf{x}^*$, is an equilibrium point of (3.7). Therefore, instead of studying the behavior of the equation (3.2) in the neighborhood of $\mathbf{x}^*$, one can equivalently study the behavior of the equations (3.7) in the neighborhood of the origin.

NOMINAL MOTION

In some practical problems, we are not concerned with stability around an equilibrium point, but rather with the stability of a *motion*, *i.e*, whether a system will remain close to its original motion trajectory if slightly perturbed away from it, as exemplified by the aircraft trajectory control problem mentioned at the beginning of this chapter. We can show that this kind of motion stability problem can be transformed into an equivalent stability problem around an equilibrium point, although the equivalent system is now non-autonomous.

Let $\mathbf{x}^*(t)$ be the solution of equation (3.2), *i.e.*, the nominal motion trajectory, corresponding to initial condition $\mathbf{x}^*(0) = \mathbf{x}_o$. Let us now perturb the initial condition to be $\mathbf{x}(0) = \mathbf{x}_o + \delta\mathbf{x}_o$ and study the associated variation of the motion error

$$\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{x}^*(t)$$

as illustrated in Figure 3.2. Since both $\mathbf{x}^*(t)$ and $\mathbf{x}(t)$ are solutions of (3.2), we have

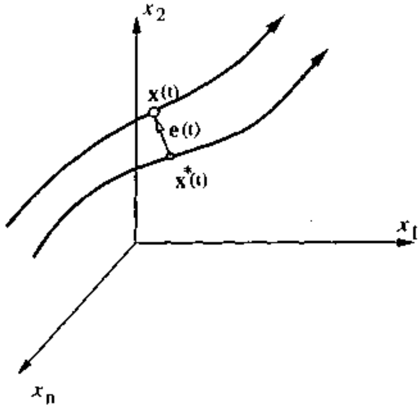


Figure 3.2 : Nominal and Perturbed Motions

$$\dot{\mathbf{x}}^* = \mathbf{f}(\mathbf{x}^*) \qquad \mathbf{x}(0) = \mathbf{x}_o$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad \mathbf{x}(0) = \mathbf{x}_o + \delta\mathbf{x}_o$$

then $\mathbf{e}(t)$ satisfies the following non-autonomous differential equation

$$\dot{\mathbf{e}} = \mathbf{f}(\mathbf{x}^* + \mathbf{e}, t) - \mathbf{f}(\mathbf{x}^*, t) = \mathbf{g}(\mathbf{e}, t) \tag{3.8}$$

with initial condition $\mathbf{e}(0) = \delta\mathbf{x}_o$. Since $\mathbf{g}(\mathbf{0}, t) = \mathbf{0}$, the new dynamic system, with $\mathbf{e}$ as state and $\mathbf{g}$ in place of $\mathbf{f}$, has an equilibrium point at the origin of the state space. Therefore, instead of studying the deviation of $\mathbf{x}(t)$ from $\mathbf{x}^*(t)$ for the original system, we may simply study the stability of the perturbation dynamics (3.8) with respect to the equilibrium point $\mathbf{0}$. Note, however, that the perturbation dynamics is non-autonomous, due to the presence of the nominal trajectory $\mathbf{x}^*(t)$ on the right-hand side. *Each* particular nominal motion of an autonomous system corresponds to an equivalent non-autonomous system, whose study requires the non-autonomous system analysis techniques to be presented in chapter 4.

Let us now illustrate this important transformation on a specific system.

Example 3.2: Consider the autonomous mass-spring system

$$m\ddot{x} + k_1 x + k_2 x^3 = 0$$

which contains a nonlinear term reflecting the hardening effect of the spring. Let us study the stability of the motion $x^*(t)$ which starts from initial position x_o.

Assume that we slightly perturb the initial position to be $x(0) = x_o + \delta x_o$. The resulting system trajectory is denoted as $x(t)$. Proceeding as before, the equivalent differential equation governing the motion error e is

$$m\ddot{e} + k_1 e + k_2 [\, e^3 + 3e^2 x^*(t) + 3e x^{*2}(t) \,] = 0$$

Clearly, this is a non-autonomous system. □

Of course, one can also show that for non-autonomous nonlinear systems, the stability problem around a nominal motion can also be transformed as a stability problem around the origin for an equivalent non-autonomous system.

Finally, note that if the original system is autonomous and *linear*, in the form (3.4), then the equivalent system is still autonomous, since it can be written

$$\dot{\mathbf{e}} = \mathbf{A}\,\mathbf{e}$$

3.2 Concepts of Stability

In the beginning of this chapter, we introduced the intuitive notion of stability as a kind of well-behavedness around a desired operating point. However, since nonlinear systems may have much more complex and exotic behavior than linear systems, the mere notion of stability is not enough to describe the essential features of their motion. A number of more refined stability concepts, such as asymptotic stability, exponential stability and global asymptotic stability, are needed. In this section, we define these stability concepts formally, for autonomous systems, and explain their practical meanings.

A few simplifying notations are defined at this point. Let $\mathbf{B}_R$ denote the spherical region (or ball) defined by $\|\mathbf{x}\| < R$ in state-space, and $\mathbf{S}_R$ the sphere itself, defined by $\|\mathbf{x}\| = R$.

STABILITY AND INSTABILITY

Let us first introduce the basic concepts of stability and instability.

Definition 3.3 *The equilibrium state* $\mathbf{x} = \mathbf{0}$ *is said to be* <u>*stable*</u> *if, for any* $R > 0$, *there exists* $r > 0$, *such that if* $\|\mathbf{x}(0)\| < r$, *then* $\|\mathbf{x}(t)\| < R$ *for all* $t \geq 0$. *Otherwise, the equilibrium point is* <u>*unstable*</u>.

Essentially, stability (also called *stability in the sense of Lyapunov*, or *Lyapunov stability*) means that the system trajectory can be kept arbitrarily close to the origin by starting sufficiently close to it. More formally, the definition states that the origin is stable, if, given that we do not want the state trajectory $\mathbf{x}(t)$ to get out of a ball of arbitrarily specified radius $\mathbf{B}_R$, a value $r(R)$ can be found such that starting the state from within the ball $\mathbf{B}_r$ at time 0 guarantees that the state will stay within the ball $\mathbf{B}_R$ thereafter. The geometrical implication of stability is indicated by curve 2 in Figure 3.3. Chapter 2 provides examples of stable equilibrium points in the case of second-order systems, such as the origin for the mass-spring system of Example 2.1, or stable nodes or foci in the local linearization of a nonlinear system.

Throughout the book, we shall use the standard mathematical abbreviation symbols:

- $\forall$ to mean "for any"
- $\exists$ for "there exists"
- $\in$ for "in the set"
- $\Rightarrow$ for "implies that"

Of course, we shall say interchangeably that A implies B, or that A is a *sufficient condition* of B, or that B is a *necessary condition* of A. If $A \Rightarrow B$ and $B \Rightarrow A$, then A and B are *equivalent*, which we shall denote by $A \Leftrightarrow B$.

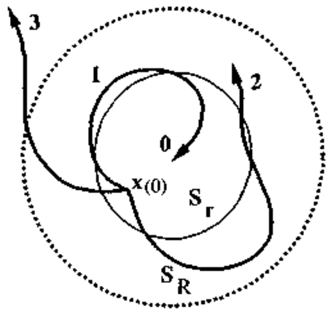
Using these symbols, Definition 3.3 can be written

$$\forall R > 0, \exists r > 0, \ \|\mathbf{x}(0)\| < r \quad \Rightarrow \quad \forall t \geq 0, \|\mathbf{x}(t)\| < R$$

or, equivalently

$$\forall R > 0, \exists r > 0, \ \mathbf{x}(0) \in \mathbf{B}_r \quad \Rightarrow \quad \forall t \geq 0, \mathbf{x}(t) \in \mathbf{B}_R$$

Conversely, an equilibrium point is unstable if there exists at least *one* ball $\mathbf{B}_R$, such that for *every* $r > 0$, no matter how small, it is always possible for the system trajectory to start somewhere within the ball $\mathbf{B}_r$ and eventually leave the ball $\mathbf{B}_R$ (Figure 3.3). Unstable nodes or saddle points in second-order systems are examples of unstable equilibria. Instability of an equilibrium point is typically undesirable, because it often leads the system into limit cycles or results in damage to the involved mechanical or electrical components.



curve 1 - asymptotically stable

curve 2 - marginally stable

curve 3 - unstable

Figure 3.3 : Concepts of stability

It is important to point out the qualitative difference between instability and the intuitive notion of "blowing up" (all trajectories close to origin move further and further away to infinity). In linear systems, instability is equivalent to blowing up, because unstable poles always lead to exponential growth of the system states. However, for nonlinear systems, blowing up is only one way of instability. The following example illustrates this point.

Example 3.3: Instability of the Van der Pol Oscillator

The Van der Pol oscillator of Example 2.6 is described by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + (1 - x_1^{\,2})x_2$$

One easily shows that the system has an equilibrium point at the origin.

As pointed out in section 2.2 and seen in the phase portrait of Figure 2.8, system trajectories starting from any non-zero initial states all asymptotically approach a limit cycle. This implies that, if we choose R in Definition 3.3 to be small enough for the circle of radius R to fall completely within the closed-curve of the limit cycle, then system trajectories starting near the origin will eventually get out of this circle (Figure 3.4). This implies instability of the origin.

Thus, even though the state of the system does remain around the equilibrium point in a certain sense, it cannot stay *arbitrarily* close to it. This is the fundamental distinction between stability and instability. □

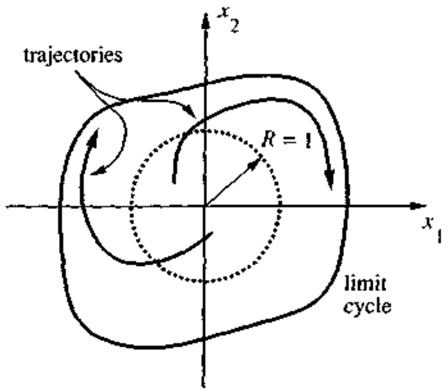


Figure 3.4 : Unstable origin of the Van der Pol Oscillator

ASYMPTOTIC STABILITY AND EXPONENTIAL STABILITY

In many engineering applications, Lyapunov stability is not enough. For example, when a satellite's attitude is disturbed from its nominal position, we not only want the satellite to maintain its attitude in a range determined by the magnitude of the disturbance, *i.e.*, Lyapunov stability, but also require that the attitude gradually go back to its original value. This type of engineering requirement is captured by the concept of *asymptotic stability*.

Definition 3.4 *An equilibrium point* $\mathbf{0}$ *is* *asymptotically stable* *if it is stable, and if in addition there exists some* $r > 0$ *such that* $\| \mathbf{x}(0) \| < r$ *implies that* $\mathbf{x}(t) \rightarrow \mathbf{0}$ *as* $t \rightarrow \infty$.

Asymptotic stability means that the equilibrium is stable, and that in addition, states started close to $\mathbf{0}$ actually converge to $\mathbf{0}$ as time t goes to infinity. Figure 3.3 shows that system trajectories starting from within the ball $\mathbf{B}_r$ converge to the origin. The ball $\mathbf{B}_r$ is called a *domain of attraction* of the equilibrium point (while *the* domain of attraction of the equilibrium point refers to the largest such region, *i.e.*, to the set of all points such that trajectories initiated at these points eventually converge to the origin). An equilibrium point which is Lyapunov stable but not asymptotically stable is called *marginally stable*.

One may question the need for the explicit stability requirement in the definition above, in view of the second condition of state convergence to the origin. However, it it easy to build counter-examples that show that state convergence does not necessarily imply stability. For instance, a simple system studied by Vinograd has trajectories of the form shown in Figure 3.5. All the trajectories starting from non-zero initial points within the unit disk first reach the curve C before converging to the origin. Thus, the origin is *unstable* in the sense of Lyapunov, despite the state convergence. Calling such a system unstable is quite reasonable, since a curve such as C may be outside the region where the model is valid – for instance, the subsonic and supersonic dynamics of a high-performance aircraft are radically different, while, with the problem under study using subsonic dynamic models, C could be in the supersonic range.

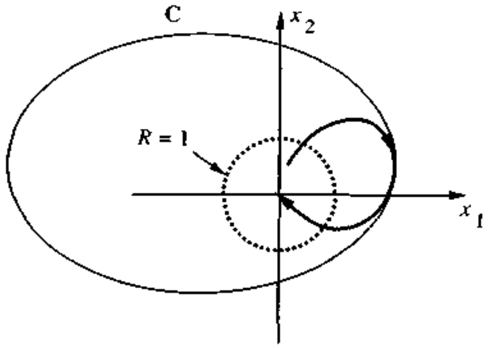


Figure 3.5 : State convergence does not imply stability

In many engineering applications, it is still not sufficient to know that a system will converge to the equilibrium point after infinite time. There is a need to estimate *how fast* the system trajectory approaches **0**. The concept of *exponential stability* can be used for this purpose.

Definition 3.5 *An equilibrium point* **0** *is* <u>*exponentially stable*</u> *if there exist two strictly positive numbers* α *and* λ *such that*

$$\forall\, t > 0, \quad \| \mathbf{x}(t) \| \le \alpha \| \mathbf{x}(0) \| \, e^{-\lambda t} \tag{3.9}$$

in some ball $\mathbf{B}_r$ *around the origin.*

In words, (3.9) means that the state vector of an exponentially stable system converges to the origin faster than an exponential function. The positive number λ is often called the *rate* of exponential convergence. For instance, the system

$$\dot{x} = -(1 + \sin^2 x)\, x$$

is exponentially convergent to $x = 0$ with a rate $\lambda = 1$. Indeed, its solution is

$$x(t) = x(0) \exp\left(-\int_o^t [1 + \sin^2(x(\tau))]\, d\tau\right)$$

and therefore

$$|x(t)| \le |x(0)|\, e^{-t}$$

Note that exponential stability implies asymptotic stability. But asymptotic stability does not guarantee exponential stability, as can be seen from the system

$$\dot{x} = -x^2, \quad x(0) = 1 \tag{3.10}$$

whose solution is $x = 1/(1+t)$, a function slower than any exponential function $e^{-\lambda t}$ (with $\lambda > 0$).

The definition of exponential convergence provides an explicit bound on the state at any time, as seen in (3.9). By writing the positive constant α as $\alpha = e^{\lambda \tau_o}$, it is easy to see that, after a time of $\tau_o + (1/\lambda)$, the magnitude of the state vector decreases to less than 35% ($\approx e^{-1}$) of its original value, similarly to the notion of *time-constant* in a linear system. After $\tau_o + (3/\lambda)$, the state magnitude $\|\mathbf{x}(t)\|$ will be less than 5% ($\approx e^{-3}$) of $\|\mathbf{x}(0)\|$.

LOCAL AND GLOBAL STABILITY

The above definitions are formulated to characterize the *local* behavior of systems, *i.e.*, how the state evolves after starting near the equilibrium point. Local properties tell little about how the system will behave when the initial state is some distance away from the equilibrium, as seen for the nonlinear system in Example 1.1. Global concepts are required for this purpose.

Definition 3.6 *If asymptotic (or exponential) stability holds for any initial states, the equilibrium point is said to be asymptotically (or exponentially) stable in the large. It is also called globally asymptotically (or exponentially) stable.*

For instance, in Example 1.2 the linearized system is globally asymptotically stable, but the original system is not. The simple system in (3.10) is also globally asymptotically stable, as can be seen from its solutions.

Linear time-invariant systems are either asymptotically stable, or marginally stable, or unstable, as can be be seen from the modal decomposition of linear system solutions; linear asymptotic stability is always global and exponential, and linear instability always implies exponential blow-up. *This explains why the refined notions of stability introduced here were not previously encountered in the study of linear systems*. They are explicitly needed only for nonlinear systems.

3.3 Linearization and Local Stability

Lyapunov's linearization method is concerned with the *local* stability of a nonlinear system. It is a formalization of the intuition that a nonlinear system should behave similarly to its linearized approximation for small range motions. Because all physical systems are inherently nonlinear, Lyapunov's linearization method serves as the fundamental *justification of using linear control techniques* in practice, *i.e.*, shows that stable design by linear control guarantees the stability of the original physical system locally.

Consider the autonomous system in (3.2), and assume that $\mathbf{f}(\mathbf{x})$ is continuously differentiable. Then the system dynamics can be written as

$$\dot{\mathbf{x}} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{0}} \mathbf{x} + \mathbf{f}_{h.o.t.}(\mathbf{x}) \tag{3.11}$$

where $\mathbf{f}_{h.o.t.}$ stands for higher-order terms in $\mathbf{x}$. Note that the above Taylor expansion starts directly with the first-order term, due to the fact that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, since $\mathbf{0}$ is an equilibrium point. Let us use the constant matrix $\mathbf{A}$ to denote the Jacobian matrix of $\mathbf{f}$ with respect to $\mathbf{x}$ at $\mathbf{x} = \mathbf{0}$ (an $n \times n$ matrix of elements $\partial f_i / \partial x_j$)

$$\mathbf{A} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{0}}$$

Then, the system

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} \tag{3.12}$$

is called the <u>*linearization*</u> (or <u>*linear approximation*</u>) of the original nonlinear system at the equilibrium point $\mathbf{0}$.

Note that, similarly, starting with a non-autonomous nonlinear system with a control input $\mathbf{u}$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

such that $\mathbf{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$, we can write

$$\dot{\mathbf{x}} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{(\mathbf{x}=\mathbf{0},\, \mathbf{u}=\mathbf{0})} \mathbf{x} + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)_{(\mathbf{x}=\mathbf{0},\, \mathbf{u}=\mathbf{0})} \mathbf{u} + \mathbf{f}_{h.o.t.}(\mathbf{x}, \mathbf{u})$$

where $\mathbf{f}_{h.o.t.}$ stands for higher-order terms in $\mathbf{x}$ and $\mathbf{u}$. Letting $\mathbf{A}$ denote the Jacobian matrix of $\mathbf{f}$ with respect to $\mathbf{x}$ at $(\mathbf{x} = \mathbf{0}, \mathbf{u} = \mathbf{0})$, and $\mathbf{B}$ denote the Jacobian matrix of $\mathbf{f}$ with respect to $\mathbf{u}$ at the same point (an $n \times m$ matrix of elements $\partial f_i / \partial u_j$, where m is the number of inputs)

$$\mathbf{A} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{(\mathbf{x}=\mathbf{0},\, \mathbf{u}=\mathbf{0})} \qquad \mathbf{B} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)_{(\mathbf{x}=\mathbf{0},\, \mathbf{u}=\mathbf{0})}$$

the system

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}$$

is the linearization (or linear approximation) of the original nonlinear system at $(\mathbf{x} = \mathbf{0}, \mathbf{u} = \mathbf{0})$.

Furthermore, the choice of a control law of the form $\mathbf{u} = \mathbf{u}(\mathbf{x})$ (with $\mathbf{u}(\mathbf{0}) = \mathbf{0}$) transforms the original non-autonomous system into an autonomous closed-loop system, having $\mathbf{x} = \mathbf{0}$ as an equilibrium point. Linearly approximating the control law as

$$\mathbf{u} \approx \left(\frac{d\mathbf{u}}{d\mathbf{x}}\right)_{\mathbf{x}=\mathbf{0}} \mathbf{x} = \mathbf{G}\,\mathbf{x}$$

the closed-loop dynamics can be linearly approximated as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) \approx (\mathbf{A} + \mathbf{B}\,\mathbf{G})\,\mathbf{x}$$

Of course, the same linear approximation can be obtained by directly considering the autonomous closed-loop system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) = \mathbf{f}_1(\mathbf{x})$$

and linearizing the function $\mathbf{f}_1$ with respect to $\mathbf{x}$, at its equilibrium point $\mathbf{x} = \mathbf{0}$.

In practice, finding a system's linearization is often most easily done simply by neglecting any term of order higher than 1 in the dynamics, as we now illustrate.

Example 3.4: Consider the system

$$\dot{x}_1 = x_2^{\,2} + x_1 \cos x_2$$

$$\dot{x}_2 = x_2 + (x_1 + 1)\,x_1 + x_1 \sin x_2$$

Its linearized approximation about $\mathbf{x} = \mathbf{0}$ is

$$\dot{x}_1 \approx 0 + x_1 \cdot 1 = x_1$$

$$\dot{x}_2 \approx x_2 + 0 + x_1 + x_1 x_2 \approx x_2 + x_1$$

The linearized system can thus be written

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{x}$$

A similar procedure can be applied for a controlled system. Consider the system

$$\ddot{x} + 4\,\dot{x}^5 + (x^2 + 1)\,u \;=\; 0$$

The system can be linearly approximated about x = 0 as

$$\ddot{x} + 0 + (0 + 1)\,u \;\approx\; 0$$

i.e., the linearized system can be written

$$\ddot{x} = -\,u$$

Assume that the control law for the original nonlinear system has been selected to be

$$u \;=\; \sin x + x^3 + \dot{x}\cos^2 x$$

then the linearized closed-loop dynamics is

$$\ddot{x} + \dot{x} + x = 0$$

□

The following result makes precise the relationship between the stability of the linear system (3.12) and that of the original nonlinear system (3.2).

Theorem 3.1 (Lyapunov's linearization method)

- *If the linearized system is strictly stable (i.e, if all eigenvalues of* **A** *are strictly in the left-half complex plane), then the equilibrium point is asymptotically stable (for the actual nonlinear system).*

- *If the linearized system is unstable (i.e, if at least one eigenvalue of* **A** *is strictly in the right-half complex plane), then the equilibrium point is unstable (for the nonlinear system).*

- *If the linearized system is marginally stable (i.e, all eigenvalues of* **A** *are in the left-half complex plane, but at least one of them is on the* $j\omega$ *axis), then one cannot conclude anything from the linear approximation (the equilibrium point may be stable, asymptotically stable, or unstable for the nonlinear system).*

While the proof of this theorem (which is actually based on Lyapunov's direct method, see Exercise 3.12) shall not be detailed, let us remark that its results are *intuitive*. A summary of the theorem is that it is true *by continuity*. If the linearized system is strictly stable, or strictly unstable, then, since the approximation is valid "not too far" from the equilibrium, the nonlinear system itself is locally stable, or locally unstable. However, if the linearized system is marginally stable, the higher-order terms in (3.11) can have a decisive effect on whether the nonlinear system is stable or unstable. As we shall see in the next section, simple nonlinear systems may be globally asymptotically stable while their linear approximations are only marginally stable: one simply cannot infer any stability property of a nonlinear system from its marginally stable linear approximation.

Example 3.5: As expected, it can be shown easily that the equilibrium points $(\theta = \pi\ [2\pi]\ ,\ \dot{\theta} = 0)$ of the pendulum of Example 3.1 are unstable. Consider for instance the equilibrium point $(\theta = \pi\ ,\ \dot{\theta} = 0)$. Since, in a neighborhood of $\theta = \pi$, we can write

$$\sin\theta \;=\; \sin\pi + \cos\pi\,(\theta - \pi) + h.o.t. \;=\; (\pi - \theta) + h.o.t.$$

thus, letting $\tilde{\theta} = \theta - \pi$, the system's linearization *about the equilibrium point* $(\theta = \pi\ ,\ \dot{\theta} = 0)$ is

$$\ddot{\tilde{\theta}} + \frac{b}{MR^2}\,\dot{\tilde{\theta}} - \frac{g}{R}\,\tilde{\theta} = 0$$

Hence the linear approximation is unstable, and therefore so is the nonlinear system at this equilibrium point. □

Example 3.6: Consider the first order system

$$\dot{x} = ax + bx^5$$

The origin 0 is one of the two equilibrium points of this system. The linearization of this system around the origin is

$$\dot{x} = ax$$

The application of Lyapunov's linearization method indicates the following stability properties of the nonlinear system

- $a < 0$: asymptotically stable;
- $a > 0$: unstable;
- $a = 0$: cannot tell from linearization.

In the third case, the nonlinear system is

$$\dot{x} = bx^5$$

The linearization method fails while, as we shall see, the direct method to be described can easily solve this problem. □

Lyapunov's linearization theorem shows that linear control design is a matter of *consistency*: one must design a controller such that the system remain in its "linear range". It also stresses major limitations of linear design: how large is the linear range? What is the *extent* of stability (how large is r in Definition 3.3) ? These questions motivate a deeper approach to the nonlinear control problem, Lyapunov's direct method.

3.4 Lyapunov's Direct Method

The basic philosophy of Lyapunov's direct method is the mathematical extension of a fundamental physical observation: if the total *energy* of a mechanical (or electrical) system is continuously dissipated, then the system, *whether linear or nonlinear*, must eventually settle down to an equilibrium point. Thus, we may conclude the stability of a system by examining the variation of a single *scalar* function.

Specifically, let us consider the nonlinear mass-damper-spring system in Figure 3.6, whose dynamic equation is

$$m\ddot{x} + b\dot{x}|\dot{x}| + k_o x + k_1 x^3 = 0 \tag{3.13}$$

with $b\dot{x}|\dot{x}|$ representing nonlinear dissipation or damping, and $(k_o x + k_1 x^3)$ representing a nonlinear spring term. Assume that the mass is pulled away from the natural length of the spring by a large distance, and then released. Will the resulting motion be stable? It is very difficult to answer this question using the definitions of stability, because the general solution of this nonlinear equation is unavailable. The linearization method cannot be used either because the motion starts outside the linear range (and in any case the system's linear approximation is only marginally stable). However, examination of the system energy can tell us a lot about the motion pattern.

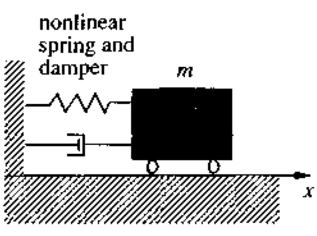


Figure 3.6 : A nonlinear mass-damper-spring system

The total mechanical energy of the system is the sum of its kinetic energy and its potential energy

$$V(\mathbf{x}) = \frac{1}{2} m\dot{x}^2 + \int_o^x (k_o x + k_1 x^3)\, dx = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} k_o x^2 + \frac{1}{4} k_1 x^4 \qquad (3.14)$$

Comparing the definitions of stability and mechanical energy, one can easily see some relations between the mechanical energy and the stability concepts described earlier:

- zero energy corresponds to the equilibrium point ($\mathbf{x} = \mathbf{0}$, $\dot{\mathbf{x}} = \mathbf{0}$)
- asymptotic stability implies the convergence of mechanical energy to zero
- instability is related to the growth of mechanical energy

These relations indicate that the value of a scalar quantity, the mechanical energy, indirectly reflects the magnitude of the state vector; and furthermore, that the stability properties of the system can be characterized by the variation of the mechanical energy of the system.

The rate of energy variation during the system's motion is obtained easily by differentiating the first equality in (3.14) and using (3.13)

$$\dot{V}(\mathbf{x}) = m\dot{x}\ddot{x} + (k_o x + k_1 x^3)\,\dot{x} = \dot{x}\,(-b\dot{x}|\dot{x}|) = -b|\dot{x}|^3 \qquad (3.15)$$

Equation (3.15) implies that the energy of the system, starting from some initial value, is continuously dissipated by the damper until the mass settles down, *i.e.*, until $\dot{x} = 0$. Physically, it is easy to see that the mass must finally settle down at the natural length of the spring, because it is subjected to a non-zero spring force at any position other than the natural length.

The direct method of Lyapunov is based on a generalization of the concepts in the above mass-spring-damper system to more complex systems. Faced with a set of nonlinear differential equations, the basic procedure of Lyapunov's direct method is to generate a scalar "energy-like" function for the dynamic system, and examine the time variation of that scalar function. In this way, conclusions may be drawn on the stability of the set of differential equations without using the difficult stability definitions or requiring explicit knowledge of solutions.

3.4.1 Positive Definite Functions and Lyapunov Functions

The energy function in (3.14) has two properties. The first is a property of the function itself: it is strictly positive unless both state variables x and $\dot{x}$ are zero. The second property is a property associated with the dynamics (3.13): the function is monotonically decreasing when the variables x and $\dot{x}$ vary according to (3.13). In Lyapunov's direct method, the first property is formalized by the notion of *positive definite functions*, and the second is formalized by the so-called Lyapunov functions.

Let us discuss positive definite functions first.

Definition 3.7 *A scalar continuous function $V(\mathbf{x})$ is said to be* <u>*locally positive definite*</u> *if $V(\mathbf{0}) = 0$ and, in a ball $\mathbf{B}_{R_o}$*

$$\mathbf{x} \neq \mathbf{0} \quad => \quad V(\mathbf{x}) > 0$$

If $V(\mathbf{0}) = 0$ and the above property holds over the whole state space, then $V(\mathbf{x})$ is said to be <u>*globally positive definite*</u>.

For instance, the function

$$V(\mathbf{x}) = \frac{1}{2} M R^2 x_2^2 + M R g (1 - \cos x_1)$$

which is the mechanical energy of the pendulum of Example 3.1, is locally positive definite. The mechanical energy (3.14) of the nonlinear mass-damper-spring system is globally positive definite. Note that, for that system, the kinetic energy $(1/2)\, m\, \dot{x}^2$ is *not* positive definite by itself, because it can equal zero for non-zero values of x.

The above definition implies that the function V has a unique minimum at the origin $\mathbf{0}$. Actually, given any function having a *unique* minimum in a certain ball, we can construct a locally positive definite function simply by adding a constant to that function. For example, the function $V(\mathbf{x}) = x_1^2 + x_2^2 - 1$ is a lower bounded function with a unique minimum at the origin, and the addition of the constant 1 to it makes it a positive definite function. Of course, the function shifted by a constant has the same time-derivative as the original function.

Let us describe the geometrical meaning of locally positive definite functions. Consider a positive definite function $V(\mathbf{x})$ of two state variables x_1 and x_2. Plotted in a 3-dimensional space, $V(\mathbf{x})$ typically corresponds to a surface looking like an upward cup (Figure 3.7). The lowest point of the cup is located at the origin.

A second geometrical representation can be made as follows. Taking x_1 and x_2 as Cartesian coordinates, the level curves $V(x_1, x_2) = V_\alpha$ typically represent a set of ovals surrounding the origin, with each oval corresponding to a positive value of V_α. These ovals, often called *contour curves*, may be thought as the sections of the cup by horizontal planes, projected on the $(x_1\,, x_2)$ plane (Figure 3.8). Note that the contour curves do not intersect, because $V(x_1, x_2)$ is uniquely defined given $(x_1\,, x_2)$.

A few related concepts can be defined similarly, in a local or global sense, *i.e.*, a function $V(\mathbf{x})$ is *negative definite* if $-V(\mathbf{x})$ is positive definite; $V(\mathbf{x})$ is *positive semi-definite* if $V(0) = 0$ and $V(\mathbf{x}) \geq 0$ for $\mathbf{x} \neq \mathbf{0}$; $V(\mathbf{x})$ is *negative semi-definite* if $-V(\mathbf{x})$ is positive semi-definite. The prefix "semi" is used to reflect the possibility of V being

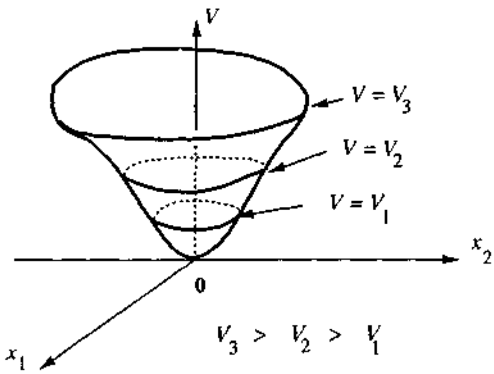


Figure 3.7 : Typical shape of a positive definite function $V(x_1, x_2)$

equal to zero for $\mathbf{x} \neq \mathbf{0}$. These concepts can be given geometrical meanings similar to the ones given for positive definite functions.

With $\mathbf{x}$ denoting the state of the system (3.2), a scalar function $V(\mathbf{x})$ actually represents an implicit function of time t. Assuming that $V(\mathbf{x})$ is differentiable, its derivative with respect to time can be found by the chain rule,

$$\dot{V} = \frac{dV(\mathbf{x})}{dt} = \frac{\partial V}{\partial \mathbf{x}}\, \dot{\mathbf{x}} = \frac{\partial V}{\partial \mathbf{x}}\, \mathbf{f}(\mathbf{x})$$

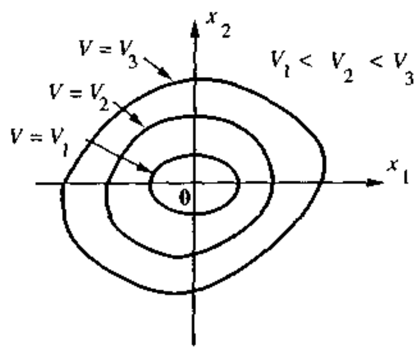


Figure 3.8 : Interpreting positive definite functions using contour curves

We see that, because $\mathbf{x}$ is required to satisfy the autonomous state equations (3.2), $\dot{V}$ only depends on $\mathbf{x}$. It is often referred to as "the derivative of V along the system trajectory" – in particular, $\dot{V} = 0$ at an equilibrium point. For the system (3.13), $\dot{V}(\mathbf{x})$ is computed in (3.15) and found to be negative. Functions such as V in that example are given a special name because of their importance in Lyapunov's direct method.

Definition 3.8 *If, in a ball* $\mathbf{B}_{R_o}$, *the function* $V(\mathbf{x})$ *is positive definite and has continuous partial derivatives, and if its time derivative along any state trajectory of system (3.2) is negative semi-definite, i.e.,*

$$\dot{V}(\mathbf{x}) \leq 0$$

then $V(\mathbf{x})$ *is said to be a* Lyapunov function *for the system (3.2).*

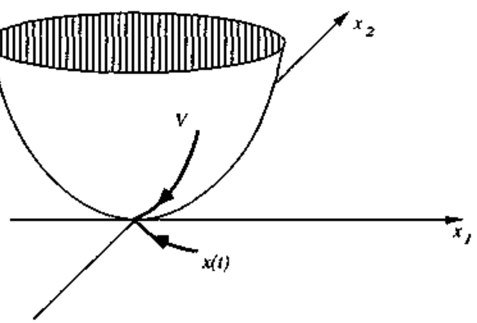


Figure 3.9 : Illustrating Definition 3.8 for $n = 2$

A Lyapunov function can be given simple geometrical interpretations. In Figure 3.9, the point denoting the value of $V(x_1, x_2)$ is seen to always point down a bowl. In Figure 3.10, the state point is seen to move across contour curves corresponding to lower and lower values of V.

3.4.2 Equilibrium Point Theorems

The relations between Lyapunov functions and the stability of systems are made precise in a number of theorems in Lyapunov's direct method. Such theorems usually have local and global versions. The local versions are concerned with stability properties in the neighborhood of equilibrium point and usually involve a locally positive definite function.

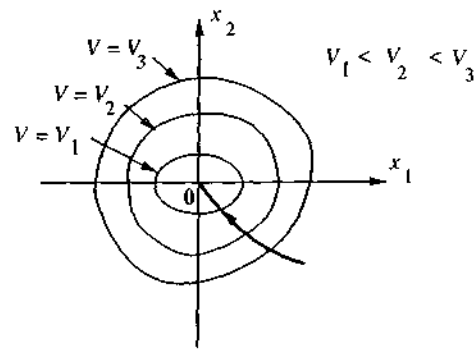


Figure 3.10 : Illustrating Definition 3.8 for $n = 2$ using contour curves

LYAPUNOV THEOREM FOR LOCAL STABILITY

Theorem 3.2 (Local Stability) *If, in a ball* $\mathbf{B}_{R_o}$ *, there exists a scalar function* $V(\mathbf{x})$ *with continuous first partial derivatives such that*

- $V(\mathbf{x})$ *is positive definite (locally in* $\mathbf{B}_{R_o}$ *)*
- $\dot{V}(\mathbf{x})$ *is negative semi-definite (locally in* $\mathbf{B}_{R_o}$ *)*

then the equilibrium point $\mathbf{0}$ *is stable. If, actually, the derivative* $\dot{V}(\mathbf{x})$ *is locally negative definite in* $\mathbf{B}_{R_o}$ *, then the stability is asymptotic.*

The proof of this fundamental result is conceptually simple, and is typical of many proofs in Lyapunov theory.

Proof: Let us derive the result using the geometric interpretation of a Lyapunov function, as illustrated in Figure 3.9 in the case $n = 2$. To show stability, we must show that given any strictly positive number R, there exists a (smaller) strictly positive number r such that any trajectory starting inside the ball $\mathbf{B}_r$ remains inside the ball $\mathbf{B}_R$ for all future time. Let m be the *minimum of* V *on the sphere* $\mathbf{S}_R$. Since V is continuous and positive definite, m exists and is strictly positive. Furthermore, since $V(\mathbf{0}) = 0$, there exists a ball $\mathbf{B}_r$ around the origin such that $V(\mathbf{x}) < m$ for any $\mathbf{x}$ inside the ball (Figure 3.11a). Consider now a trajectory whose initial point $\mathbf{x}(0)$ is within the ball $\mathbf{B}_r$. Since V is non-increasing along system trajectories, *V remains strictly smaller than m,* and therefore the trajectory cannot possibly cross the outside sphere $\mathbf{S}_R$. Thus, any trajectory starting inside the ball $\mathbf{B}_r$ remains inside the ball $\mathbf{B}_R$, and therefore Lyapunov stability is guaranteed.

Let us now assume that $\dot{V}$ is negative definite, and show asymptotic stability, by contradiction. Consider a trajectory starting in some ball $\mathbf{B}_r$ as constructed above (*e.g.*, the ball $\mathbf{B}_r$

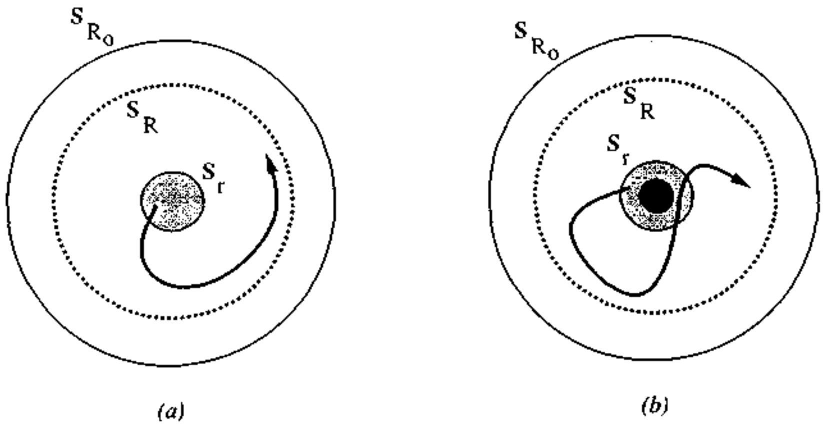


Figure 3.11 : Illustrating the proof of Theorem 3.2 for $n = 2$

corresponding to $R = R_o$). Then the trajectory will remain in the ball $\mathbf{B}_R$ for all future time. Since V is lower bounded and decreases continually, V tends towards a limit L, such that $\forall\, t \geq 0$, $V(\mathbf{x}(t)) \geq L$. Assume that this limit is not zero, *i.e.*, that $L > 0$. Then, since V is continuous and $V(\mathbf{0}) = 0$, there exists a ball $\mathbf{B}_{r_o}$ that the system trajectory never enters (Figure 3.11b). But then, *since* $-\dot{V}$ *is also continuous and positive definite*, and since $\mathbf{B}_R$ is bounded, $-\dot{V}$ must remain larger than some strictly positive number L_1. This is a contradiction, because it would imply that $V(t)$ decreases from its initial value V_o to a value strictly smaller than L, in a finite time smaller than $[V_o - L]/L_1$. Hence, all trajectories starting in $\mathbf{B}_r$ asymptotically converge to the origin. □

In applying the above theorem for analysis of a nonlinear system, one goes through the two steps of choosing a positive definite function, and then determining its derivative along the path of the nonlinear systems. The following example illustrates this procedure.

Example 3.7: Local Stability

A simple pendulum with viscous damping is described by

$$\ddot{\theta} + \dot{\theta} + \sin\theta = 0$$

Consider the following scalar function

$$V(\mathbf{x}) = (1 - \cos\theta) + \frac{\dot{\theta}^2}{2}$$

One easily verifies that this function is locally positive definite. As a matter of fact, this function represents the total energy of the pendulum, composed of the sum of the potential energy and the kinetic energy. Its time-derivative is easily found to be

$$\dot{V}(\mathbf{x}) = \dot{\theta}\sin\theta + \dot{\theta}\ddot{\theta} = -\dot{\theta}^2 \le 0$$

Therefore, by invoking the above theorem, one concludes that the origin is a stable equilibrium point. In fact, using physical insight, one easily sees the reason why $\dot{V}(\mathbf{x}) \le 0$, namely that the damping term absorbs energy. Actually, $\dot{V}$ is precisely the power dissipated in the pendulum. However, with this Lyapunov function, one cannot draw conclusions on the asymptotic stability of the system, because $\dot{V}(\mathbf{x})$ is only negative semi-definite. □

The following example illustrates the asymptotic stability result.

Example 3.8: Asymptotic stability

Let us study the stability of the nonlinear system defined by

$$\dot{x}_1 = x_1\,(x_1^{\,2} + x_2^{\,2} - 2) - 4\,x_1 x_2^{\,2}$$

$$\dot{x}_2 = 4\,x_1^{\,2} x_2 + x_2\,(x_1^{\,2} + x_2^{\,2} - 2)$$

around its equilibrium point at the origin. Given the positive definite function

$$V(x_1\,, x_2) = x_1^{\,2} + x_2^{\,2}$$

its derivative $\dot{V}$ along any system trajectory is

$$\dot{V} = 2\,(x_1^{\,2} + x_2^{\,2})\,(x_1^{\,2} + x_2^{\,2} - 2\,)$$

Thus, $\dot{V}$ is locally negative definite in the 2-dimensional ball $\mathbf{B}_2$, *i.e.*, in the region defined by $x_1^{\,2} + x_2^{\,2} < 2$. Therefore, the above theorem indicates that the origin is asymptotically stable. □

LYAPUNOV THEOREM FOR GLOBAL STABILITY

The above theorem applies to the local analysis of stability. In order to assert *global asymptotic stability* of a system, one might naturally expect that the ball $\mathbf{B}_{R_o}$ in the above local theorem has to be expanded to be the whole state-space. This is indeed necessary, but it is not enough. An additional condition on the function V has to be satisfied: $V(\mathbf{x})$ must be *radially unbounded*, by which we mean that $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$ (in other words, as $\mathbf{x}$ tends to infinity in *any* direction). We then obtain the following powerful result:

Theorem 3.3 (Global Stability) *Assume that there exists a scalar function V of the state* $\mathbf{x}$, *with continuous first order derivatives such that*

- $V(\mathbf{x})$ *is positive definite*
- $\dot{V}(\mathbf{x})$ *is negative definite*
- $V(\mathbf{x}) \to \infty \quad as \quad \|\mathbf{x}\| \to \infty$

then the equilibrium at the origin is globally asymptotically stable.

Proof: The proof is the same as in the local case, by noticing that the radial unboundedness of V, combined with the negative definiteness of $\dot{V}$, implies that, given any initial condition $\mathbf{x}_o$, the trajectories remain in the *bounded* region defined by $V(\mathbf{x}) \leq V(\mathbf{x}_o)$. □

The reason for the radial unboundedness condition is to assure that the contour curves (or contour surfaces in the case of higher order systems) $V(\mathbf{x}) = V_\alpha$ correspond to closed curves. If the curves are not closed, it is possible for the state trajectories to drift away from the equilibrium point, even though the state keeps going through contours corresponding to smaller and smaller V_α's. For example, for the positive definite function $V = [\, x_1^2/(1 + x_1^2)] + x_2^2$, the curves $V(\mathbf{x}) = V_\alpha$ for $V_\alpha > 1$ are open curves. Figure 3.12 shows the divergence of the state while moving toward lower and lower "energy" curves. Exercise 3.4 further illustrates this point on a specific system.

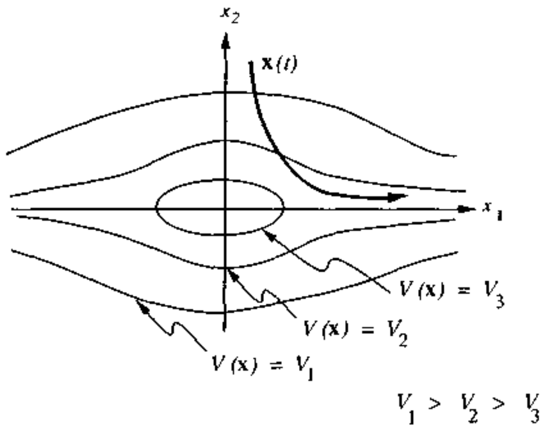


Figure 3.12 : Motivation of the radial unboundedness condition

Example 3.9: A class of first-order systems

Consider the nonlinear system

$$\dot{x} + c(x) = 0$$

where c *is any continuous function* of the same sign as its scalar argument x, *i.e.*,

$$x\, c(x) > 0 \quad \text{for} \quad x \neq 0$$

Intuitively, this condition indicates that $-c(x)$ "pushes" the system back towards its rest position $x = 0$, but is otherwise arbitrary. Since c is continuous, it also implies that $c(0) = 0$ (Figure 3.13).

Consider as the Lyapunov function candidate the square of the distance to the origin

$$V = x^2$$

The function V is radially unbounded, since it tends to infinity as $|x| \to \infty$. Its derivative is

$$\dot{V} = 2\, x\, \dot{x} = = -\, 2\, x\, c(x)$$

Thus $\dot{V} < 0$ as long as $x \neq 0$, so that $x = 0$ is a globally asymptotically stable equilibrium point.

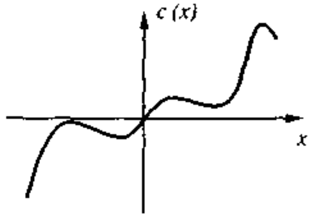


Figure 3.13 : The function $c(x)$

For instance, the system

$$\dot{x} = \sin^2 x - x$$

is globally asymptotically convergent to $x = 0$, *since for* $x \neq 0$, $\sin^2 x \leq |\sin x| < |x|$. Similarly, the system

$$\dot{x} = -\, x^3$$

is globally asymptotically convergent to $x = 0$. Notice that while this system's linear approximation ($\dot{x} \approx 0$) is inconclusive, even about local stability, the actual nonlinear system enjoys a strong stability property (*global asymptotic* stability). □

Example 3.10: Consider the system

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2)$$

$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2)$$

The origin of the state-space is an equilibrium point for this system. Let V be the positive definite function

$$V(\mathbf{x}) = x_1^2 + x_2^2$$

The derivative of V along any system trajectory is

$$\dot{V}(\mathbf{x}) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = -2(x_1^2 + x_2^2)^2$$

which is negative definite. Therefore, the origin is a globally asymptotically stable equilibrium point. Note that the globalness of this stability result also implies that the origin is the *only* equilibrium point of the system. □

REMARKS

Many Lyapunov functions may exist for the same system. For instance, if V is a Lyapunov function for a given system, so is

$$V_1 = \rho V^{\alpha}$$

where ρ is any strictly positive constant and α is any scalar (not necessarily an integer) larger than 1. Indeed, the positive-definiteness of V implies that of V_1, the positive-definiteness (or positive semi-definiteness) of $-\dot{V}$ implies that of $-\dot{V}_1$, and (the radial unboundedness of V (if applicable) implies that of V_1.

More importantly, for a given system, specific choices of Lyapunov functions may yield more precise results than others. Consider again the pendulum of Example 3.7. The function

$$V(\mathbf{x}) = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}(\dot{\theta} + \theta)^2 + 2(1 - \cos\theta)$$

is also a Lyapunov function for the system, because locally

$$\dot{V}(\mathbf{x}) = -(\dot{\theta}^2 + \theta\sin\theta) \leq 0$$

However, it is interesting to note that $\dot{V}$ is actually locally *negative definite*, and therefore, this modified choice of V, without obvious physical meaning, allows the asymptotic stability of the pendulum to be shown.

Along the same lines, it is important to realize that the theorems in Lyapunov analysis are all *sufficiency* theorems. If for a particular choice of Lyapunov function candidate V, the conditions on $\dot{V}$ are not met, one cannot draw any conclusions on the stability or instability of the system – the only conclusion one should draw is that a different Lyapunov function candidate should be tried.

3.4.3 Invariant Set Theorems

Asymptotic stability of a control system is usually a very important property to be determined. However, the equilibrium point theorems just described are often difficult to apply in order to assert this property. The reason is that it often happens that $\dot{V}$, the derivative of the Lyapunov function candidate, is only negative *semi*-definite, as seen in (3.15). In this kind of situation, fortunately, it is still possible to draw conclusions on asymptotic stability, with the help of the powerful *invariant set theorems*, attributed to La Salle. This section presents the local and global versions of the invariant set theorems.

The central concept in these theorems is that of invariant set, a generalization of the concept of equilibrium point.

Definition 3.9 *A set* **G** *is an* <u>*invariant set*</u> *for a dynamic system if every system trajectory which starts from a point in* **G** *remains in* **G** *for all future time.*

For instance, any equilibrium point is an invariant set. The domain of attraction of an equilibrium point is also an invariant set. A trivial invariant set is the whole state-space. For an autonomous system, any of the trajectories in state-space is an invariant set. Since limit cycles are special cases of system trajectories (closed curves in the phase plane), they are also invariant sets.

Besides often yielding conclusions on asymptotic stability when $\dot{V}$, the derivative of the Lyapunov function candidate, is only negative semi-definite, the invariant set theorems also allow us to *extend* the concept of Lyapunov function so as to describe convergence to dynamic behaviors more general than equilibrium, *e.g.*, convergence to a limit cycle.

Similarly to our earlier discussion of Lyapunov's direct method, we first discuss the local version of the invariant set theorems, and then the global version.

LOCAL INVARIANT SET THEOREM

The invariant set theorems reflect the intuition that the decrease of a Lyapunov function V has to gradually vanish (*i.e.*, $\dot{V}$ has to converge to zero) because V is lower bounded. A precise statement of this result is as follows.

Theorem 3.4 (Local Invariant Set Theorem) *Consider an autonomous system of the form (3.2), with* $\mathbf{f}$ *continuous, and let* $V(\mathbf{x})$ *be a scalar function with continuous first partial derivatives. Assume that*

- *for some* $l > 0$*, the region* Ω_l *defined by* $V(\mathbf{x}) < l$ *is bounded*
- $\dot{V}(\mathbf{x}) \leq 0$ *for all* $\mathbf{x}$ *in* Ω_l

Let $\mathbf{R}$ *be the set of all points within* Ω_l *where* $\dot{V}(\mathbf{x}) = 0$*, and* $\mathbf{M}$ *be the largest invariant set in* $\mathbf{R}$*. Then, every solution* $\mathbf{x}(t)$ *originating in* Ω_l *tends to* $\mathbf{M}$ *as* $t \to \infty$.

In the above theorem, the word "largest" is understood in the sense of set theory, *i.e.*, $\mathbf{M}$ is the *union* of all invariant sets (*e.g.*, equilibrium points or limit cycles) within $\mathbf{R}$. In particular, if the set $\mathbf{R}$ is itself invariant (*i.e.*, if once $\dot{V} = 0$, then $\dot{V} \equiv 0$ for all future time), then $\mathbf{M} = \mathbf{R}$. Also note that V, although often still referred to as a Lyapunov function, is not required to be positive definite.

The geometrical meaning of the theorem is illustrated in Figure 3.14, where a trajectory starting from within the bounded region Ω_l is seen to converge to the largest invariant set $\mathbf{M}$. Note that the set $\mathbf{R}$ is not necessarily connected, nor is the set $\mathbf{M}$.

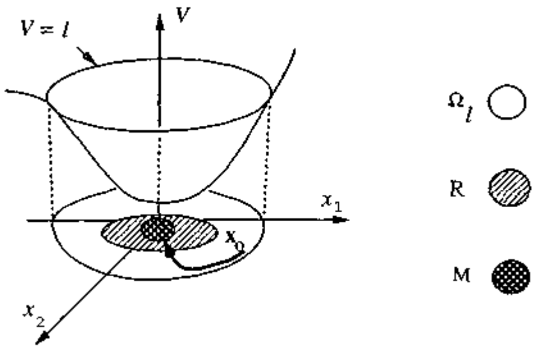


Figure 3.14 : Convergence to the largest invariant set $\mathbf{M}$

The theorem can be proven in two steps, by first showing that $\dot{V}$ goes to zero, and then showing that the state converges to the largest invariant set within the set defined by $\dot{V} = 0$. We shall simply give a sketch of the proof, since the detailed proof of the second part involves a number of concepts in topology and real analysis which are not prerequisites of this text.

Proof: The first part of the proof involves showing that $\dot{V} \to 0$ for any trajectory starting from a point in Ω_l, using a result in functional analysis known as Barbalat's lemma, which we shall detail in section 4.3.

Specifically, consider a trajectory starting from an arbitrary point $\mathbf{x}_o$ in Ω_l. The trajectory must stay in Ω_l all the time, because $\dot{V} \leq 0$ implies that $V[\mathbf{x}(t)] \leq V[\mathbf{x}(0)] < l$ for all $t \geq 0$. In addition, because $V(\mathbf{x})$ is continuous in $\mathbf{x}$ (since it is differentiable with respect to $\mathbf{x}$) over the bounded region Ω_l, it is lower bounded in that region; therefore, since we just noticed that the trajectory remains in Ω_l, $V[\mathbf{x}(t)]$ remains lower bounded for all $t \geq 0$. Furthermore, the facts that $\mathbf{f}$ is continuous, V has continuous partial derivatives, and the region Ω_l is bounded, imply that $\dot{V}$ is uniformly continuous. Therefore, $V[\mathbf{x}(t)]$ satisfies the three conditions (V lower bounded; $\dot{V} \leq 0$; $\dot{V}$ uniformly continuous) of Barbalat's lemma. As a result, $\dot{V}[\mathbf{x}(t)] \to 0$, which implies that all system trajectories starting from within Ω_l converge to the set $\mathbf{R}$.

The second part of the proof [see, *e.g.*, Hahn, 1968] involves showing that the trajectories cannot converge to just anywhere in the set $\mathbf{R}$: they must converge to the largest invariant set $\mathbf{M}$ within $\mathbf{R}$. This can be proven by showing that any bounded trajectory of an autonomous system converges to an invariant set (the so-called positive limit set of the trajectory), and then simply noticing that this set is a subset of the largest invariant set $\mathbf{M}$. □

Note that the asymptotic stability result in the local Lyapunov theorem can be viewed a special case of the above invariant set theorem, where the set $\mathbf{M}$ consists only of the origin.

Let us now illustrate applications of the invariant set theorem using some examples. The first example shows how to conclude asymptotic stability for problems which elude the local Lyapunov theorem. The second example shows how to determine a *domain of attraction*, an issue which was not specifically addressed before. The third example shows the convergence of system trajectories to a limit cycle.

Example 3.11: Asymptotic stability of the mass-damper-spring system

For the system (3.13), one can only draw conclusion of marginal stability using the energy function (3.14) in the local equilibrium point theorem, because $\dot{V}$ is only negative semi-definite according to (3.15). Using the invariant set theorem, however, we can show that the system is actually asymptotically stable. To do this, we only have to show that the set $\mathbf{M}$ contains only one point.

The set $\mathbf{R}$ is defined by $\dot{x} = 0$, *i.e.*, the collection of states with zero velocity, or the whole horizontal axis in the phase plane $(x, \dot{x})$. Let us show that the largest invariant set $\mathbf{M}$ in this set $\mathbf{R}$ contains only the origin. Assume that $\mathbf{M}$ contains a point with a nonzero position x_1, then, the acceleration at that point is $\ddot{x} = -(k_o/m)x - (k_1/m)x^3 \neq 0$. This implies that the trajectory will immediately move out of the set **R** and thus also out of the set **M**, a contradiction to the definition. □

Example 3.12: Domain of Attraction

Consider again the system in Example 3.8. For $l = 2$, the region Ω_2, defined by $V(\mathbf{x}) = x_1^2 + x_2^2 < 2$, is bounded. The set **R** is simply the origin **0**, which is an invariant set (since it is an equilibrium point). All the conditions of the local invariant set theorem are satisfied and, therefore, any trajectory starting within the circle converges to the origin. Thus, a domain of attraction is explicitly determined by the invariant set theorem. □

Example 3.13: Attractive Limit Cycle

Consider the system

$$\dot{x}_1 = x_2 - x_1^7\,[\,x_1^4 + 2\,x_2^2 - 10\,]$$

$$\dot{x}_2 = -\,x_1^3 - 3\,x_2^5\,[\,x_1^4 + 2\,x_2^2 - 10\,]$$

Notice first that the set defined by $x_1^4 + 2\,x_2^2 = 10$ is invariant, since

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_1^4 + 2\,x_2^2 - 10) = -\,(4\,x_1^{10} + 12\,x_2^6)\,(x_1^4 + 2\,x_2^2 - 10)$$

which is zero on the set. The motion *on* this invariant set is described (equivalently) by *either* of the equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\,x_1^3$$

Therefore, we see that the invariant set actually represents a *limit cycle*, along which the state vector moves clockwise.

Is this limit cycle actually attractive? Let us define as a Lyapunov function candidate

$$V = (x_1^4 + 2\,x_2^2 - 10)^2$$

which represents a measure of the "distance" to the limit cycle. For any arbitrary positive number l, the region Ω_l, which surrounds the limit cycle, is bounded. Using our earlier calculation, we immediately obtain

$$\dot{V} = -\,8\,(x_1^{10} + 3\,x_2^6)\,(x_1^4 + 2\,x_2^2 - 10)^2$$

Thus $\dot{V}$ is strictly negative, except if

$$x_1^4 + 2\,x_2^2 = 10 \qquad \text{or} \qquad x_1^{10} + 3\,x_2^6 = 0$$

in which case $\dot{V} = 0$. The first equation is simply that defining the limit cycle, while the second equation is verified only at the origin. Since both the limit cycle and the origin are invariant sets, the set **M** simply consists of their union. Thus, all system trajectories starting in Ω_l converge either to the limit cycle, or to the origin (Figure 3.15).

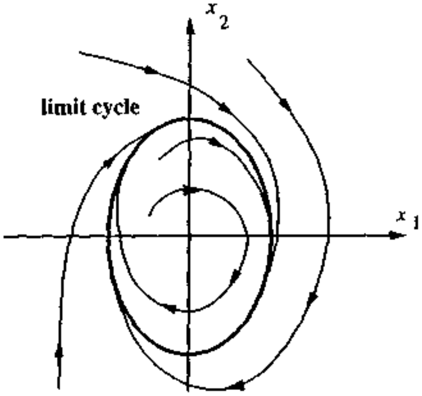


Figure 3.15 : Convergence to a limit cycle

Moreover, the equilibrium point at the origin can actually be shown to be *unstable*. However, this result cannot be obtained from linearization, since the linearized system ($\dot{x}_1 = x_2$, $\dot{x}_2 = 0$) is only marginally stable. Instead, and more astutely, consider the region Ω_{100} , and note that while the origin $\mathbf{0}$ does not belong to Ω_{100} , every other point in the region enclosed by the limit cycle is in Ω_{100} (in other words, the origin corresponds to a local *maximum* of V). Thus, while the expression of $\dot{V}$ is the same as before, *now the set* **M** *is just the limit cycle*. Therefore, reapplication of the invariant set theorem shows that any state trajectory starting from the region within the limit cycle, excluding the origin, actually converges to the limit cycle. In particular, this implies that the equilibrium point at the origin is unstable. □

Example 3.11 actually represents a very common application of the invariant set theorem: conclude asymptotic stability of an equilibrium point for systems with negative *semi*-definite $\dot{V}$. The following corollary of the invariant set theorem is more specifically tailored to such applications:

Corollary *Consider the autonomous system (3.2), with* **f** *continuous, and let* $V(\mathbf{x})$ *be a scalar function with continuous partial derivatives. Assume that in a certain neighborhood* Ω *of the origin*

- *$V(\mathbf{x})$ is locally positive definite*

- *$\dot{V}$ is negative semi-definite*

- *the set* **R** *defined by $\dot{V}(\mathbf{x}) = 0$ contains no trajectories of (3.2) other than the trivial trajectory $\mathbf{x} \equiv \mathbf{0}$*

Then, the equilibrium point **0** *is asymptotically stable. Furthermore, the largest connected region of the form $\mathbf{\Omega}_l$ (defined by $V(\mathbf{x}) < l$) within $\mathbf{\Omega}$ is a domain of attraction of the equilibrium point.*

Indeed, the largest invariant set **M** in **R** then contains only the equilibrium point **0**. Note that

- The above corollary replaces the negative definiteness condition on $\dot{V}$ in Lyapunov's local asymptotic stability theorem by a negative *semi*-definiteness condition on $\dot{V}$, combined with a third condition on the trajectories within **R**.

- The largest connected region of the form $\mathbf{\Omega}_l$ within $\mathbf{\Omega}$ is *a* domain of attraction of the equilibrium point, but not necessarily *the* whole domain of attraction, because the function V is not unique.

- The set $\mathbf{\Omega}$ itself is not necessarily a domain of attraction. Actually, the above theorem does not guarantee that $\mathbf{\Omega}$ is invariant: some trajectories starting in $\mathbf{\Omega}$ but outside of the largest $\mathbf{\Omega}_l$ may actually end up outside $\mathbf{\Omega}$.

GLOBAL INVARIANT SET THEOREMS

The above invariant set theorem and its corollary can be simply extended to a global result, by requiring the radial unboundedness of the scalar function V rather than the existence of a bounded $\mathbf{\Omega}_l$.

Theorem 3.5 (Global Invariant Set Theorem) *Consider the autonomous system (3.2), with* **f** *continuous, and let $V(\mathbf{x})$ be a scalar function with continuous first partial derivatives. Assume that*

- *$V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$*

- *$\dot{V}(\mathbf{x}) \leq 0$ over the whole state space*

Let **R** *be the set of all points where $\dot{V}(\mathbf{x}) = 0$, and* **M** *be the largest invariant set in* **R**. *Then all solutions globally asymptotically converge to* **M** *as $t \to \infty$.*

For instance, the above theorem shows that the limit cycle convergence in

Example 3.13 is actually global: all system trajectories converge to the limit cycle (unless they start exactly at the origin, which is an unstable equilibrium point).

Because of the importance of this theorem, let us present an additional (and very useful) example.

Example 3.14: A class of second-order nonlinear systems

Consider a second-order system of the form

$$\ddot{x} + b(\dot{x}) + c(x) = 0$$

where b and c are continuous functions verifying the sign conditions

$$\dot{x}\, b(\dot{x}) > 0 \quad \text{for } \dot{x} \neq 0$$

$$x\, c(x) > 0 \quad \text{for } x \neq 0$$

The dynamics of a mass-damper-spring system with nonlinear damper and spring can be described by equations of this form, with the above sign conditions simply indicating that the otherwise arbitrary functions b and c actually represent "damping" and "spring" effects. A nonlinear R-L-C (resistor-inductor-capacitor) electrical circuit can also be represented by the above dynamic equation (Figure 3.16). Note that if the functions b and c are actually *linear* ($b(\dot{x}) = \alpha_1 \dot{x}$, $c(x) = \alpha_o x$), the above sign conditions are simply the *necessary and sufficient* conditions for the system's stability (since they are equivalent to the conditions $\alpha_1 > 0, \alpha_o > 0$).

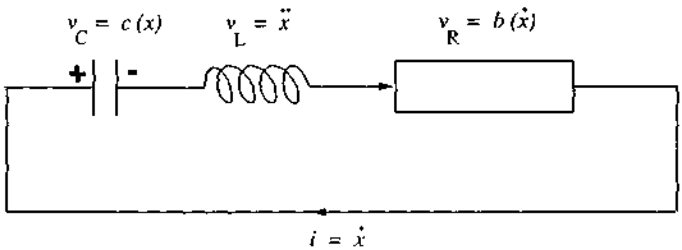


Figure 3.16 : A nonlinear R-L-C circuit

Together with the continuity assumptions, the sign conditions on the functions b and c imply that $b(0) = 0$ and $c(0) = 0$ (Figure 3.17). A positive definite function for this system is

$$V = \frac{1}{2}\dot{x}^2 + \int_o^x c(y)\, dy$$

which can be thought of as the sum of the kinetic and potential energy of the system.

Differentiating V, we obtain

$$\dot{V} = \dot{x}\,\ddot{x} + c(x)\,\dot{x} = -\dot{x}\,b(\dot{x}) - \dot{x}\,c(x) + c(x)\,\dot{x} = -\dot{x}\,b(\dot{x}) \leq 0$$

which can be thought of as representing the power dissipated in the system. Furthermore, by hypothesis, $\dot{x}\,b(\dot{x}) = 0$ only if $\dot{x} = 0$. Now $\dot{x} = 0$ implies that

$$\ddot{x} = -c(x)$$

which is nonzero as long as $x \neq 0$. Thus the system cannot get "stuck" at an equilibrium value other than $x = 0$; in other words, with **R** being the set defined by $\dot{x} = 0$, the largest invariant set **M** in **R** contains only one point, namely $[x = 0, \dot{x} = 0]$. Use of the local invariant set theorem indicates that the origin is a locally asymptotically stable point.

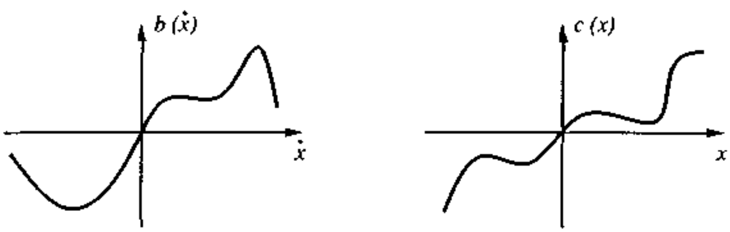


Figure 3.17 : The functions $b(\dot{x})$ and $c(x)$

Furthermore, if the integral $\int_0^x c(r)\,dr$ is unbounded as $|x| \to \infty$, then V is a radially unbounded function and the equilibrium point at the origin is globally asymptotically stable, according to the global invariant set theorem.

For instance, the system

$$\ddot{x} + \dot{x}^3 + x^5 = x^4 \sin^2 x$$

is globally asymptotically convergent to $x = 0$ (while, again, its linear approximation would be inconclusive, even about its local stability). □

The relaxation of the positive definiteness requirement on the function V, as compared with Lyapunov's direct method, also allows one to use a single Lyapunov-like function to describe systems with multiple equilibria.

Example 3.15: Multimodal Lyapunov Function

Consider the system

$$\ddot{x} + |x^2 - 1|\,\dot{x}^3 + x = \sin\frac{\pi x}{2}$$

For this system, we study, similarly to Example 3.14, the Lyapunov function

$$V = \frac{1}{2}\dot{x}^2 + \int_0^x (y - \sin\frac{\pi y}{2})\, dy$$

This function has two minima, at $x = \pm 1$; $\dot{x} = 0$, and a local *maximum* in x (a saddle point in the state-space) at $x = 0$; $\dot{x} = 0$. As in Example 3.14, the time-derivative of V is (without calculations)

$$\dot{V} = -\ |x^2 - 1|\,\dot{x}^4$$

i.e, the virtual power "dissipated" by the system. Now

$$\dot{V} = 0 \quad => \quad \dot{x} = 0 \ \text{ or } \ x = \pm 1$$

Let us consider each of these cases:

$$\dot{x} = 0 \quad => \quad \ddot{x} \ = \ \sin\frac{\pi x}{2} - x \neq 0 \ \text{ except if } \ x = 0 \text{ or } x = \pm 1$$

$$x = \pm 1 \quad => \quad \ddot{x} = 0$$

Thus, the invariant set theorem indicates that the system converges globally to ($x = 1$; $\dot{x} = 0$) or ($x = -1$; $\dot{x} = 0$), or to ($x = 0$; $\dot{x} = 0$). The first two of these equilibrium points are stable, since they correspond to local mimina of V (note again that linearization is inconclusive about their stability). By contrast, the equilibrium point ($x = 0$; $\dot{x} = 0$) is unstable, as can be shown from linearization ($\ddot{x} = (\pi/2 - 1)\,x$), or simply by noticing that because that point is a local maximum of V along the x axis, any small deviation in the x direction will drive the trajectory away from it. □

As noticed earlier, several Lyapunov functions may exist for a given system, and therefore several associated invariant sets may be derived. The system then converges to the (necessarily non-empty) *intersection* of the invariant sets $\mathbf{M}_i$, which may give a more precise result than that obtained from any of the Lyapunov functions taken separately. Equivalently, one can notice that the sum of two Lyapunov functions for a given system is also a Lyapunov function, whose set $\mathbf{R}$ is the intersection of the individual sets $\mathbf{R}_i$.

3.5 System Analysis Based on Lyapunov's Direct Method

With so many theorems and so many examples presented in the last section, one may feel confident enough to attack practical nonlinear control problems. However, the theorems all make a basic assumption: an explicit Lyapunov function is somehow known. The question is therefore how to find a Lyapunov function for a specific problem. Yet, there is no general way of finding Lyapunov functions for nonlinear systems. This is a fundamental drawback of the direct method. Therefore, faced with specific systems, one has to use experience, intuition, and physical insights to search for an appropriate Lyapunov function. In this section, we discuss a number of techniques which can facilitate the otherwise blind search of Lyapunov functions.

We first show that, not surprisingly, Lyapunov functions can be systematically found to describe stable *linear* systems. Next, we discuss two of many mathematical methods that may be used to help finding a Lyapunov function for a given nonlinear system. We then consider the use of physical insights, which, when applicable, represents by far the most powerful and elegant way of approaching the problem, and is closest in spirit to the original intuition underlying the direct method. Finally, we discuss the use of Lyapunov functions in transient performance analysis.

3.5.1 Lyapunov Analysis of Linear Time-Invariant Systems

Stability analysis for linear time-invariant systems is well known. It is interesting, however, to develop Lyapunov functions for such systems. First, this allows us to describe both linear and nonlinear systems using *a common language*, allowing shared insights between the two classes. Second, as we shall detail later on, Lyapunov functions are "additive", like energy. In other words, Lyapunov functions for combinations of subsystems may be derived by adding the Lyapunov functions of the subsystems. Since nonlinear control systems may include linear components (whether in plant or in controller), we should be able to describe linear systems in the Lyapunov formalism.

We first review some basic results on matrix algebra, since the development of Lyapunov functions for linear systems will make extensive use of quadratic forms.

SYMMETRIC, SKEW-SYMMETRIC, AND POSITIVE DEFINITE MATRICES

Definition 3.10 *A square matrix* $\mathbf{M}$ *is symmetric if* $\mathbf{M} = \mathbf{M}^T$ *(in other words, if* $\forall\, i, j \;\; M_{ij} = M_{ji}$*). A square matrix* $\mathbf{M}$ *is skew-symmetric if* $\mathbf{M} = -\mathbf{M}^T$ *(i.e. if* $\forall\, i, j \;\; M_{ij} = -M_{ji}$*).*

An interesting fact is that any square $n \times n$ matrix $\mathbf{M}$ can be represented as the sum of a symmetric matrix and a skew-symmetric matrix. This can be shown by the following decomposition

$$\mathbf{M} = \frac{\mathbf{M} + \mathbf{M}^T}{2} + \frac{\mathbf{M} - \mathbf{M}^T}{2}$$

where the first term on the left side is symmetric and the second term is skew-symmetric.

Another interesting fact is that the quadratic function associated with a skew-symmetric matrix is always zero. Specifically, let $\mathbf{M}$ be a $n \times n$ skew-symmetric matrix and $\mathbf{x}$ an arbitrary $n \times 1$ vector. Then the definition of a skew-symmetric matrix implies that

$$\mathbf{x}^T \mathbf{M} \mathbf{x} = -\, \mathbf{x}^T \mathbf{M}^T \mathbf{x}$$

Since $\mathbf{x}^T \mathbf{M}^T \mathbf{x}$ is a scalar, the right-hand side of the above equation can be replaced by its transpose. Therefore,

$$\mathbf{x}^T \mathbf{M} \mathbf{x} = -\, \mathbf{x}^T \mathbf{M} \mathbf{x}$$

This shows that

$$\forall\, \mathbf{x}\,,\ \ \mathbf{x}^T \mathbf{M} \mathbf{x} = 0 \tag{3.16}$$

In designing some tracking control systems for robots, for instance, this fact is very useful because it can simplify the control law, as we shall see in chapter 9.

Actually, property (3.16) is a *necessary and sufficient* condition for a matrix $\mathbf{M}$ to be skew-symmetric. This can be easily seen by applying (3.16) to the basis vectors $\mathbf{e}_i$:

$$[\ \forall\, i\,,\ \mathbf{e}_i^T \mathbf{M}_s \mathbf{e}_i = 0\ ] \ \ => \ \ [\ \forall\, i,\ M_{ii} = 0\ ]$$

and

$$[\ \forall\,(i,j)\,,\ (\mathbf{e}_i + \mathbf{e}_j)^T \mathbf{M}_s (\mathbf{e}_i + \mathbf{e}_j) = 0\ ] \ \ => \ \ [\ \forall\,(i,j),\ M_{ii} + M_{ij} + M_{ji} + M_{jj} = 0\ ]$$

which, using the first result, implies that

$$\forall\,(i,j)\,,\ \ M_{ji} = -\, M_{ij}$$

In our later analysis of linear systems, we will often use quadratic functions of the form $\mathbf{x}^T \mathbf{M} \mathbf{x}$ as Lyapunov function candidates. In view of the above, each quadratic function of this form, whether $\mathbf{M}$ is symmetric or not, is always equal to a quadratic function with a symmetric matrix. Thus, in considering quadratic functions of the form $\mathbf{x}^T \mathbf{M} \mathbf{x}$ as Lyapunov function candidates, one can always assume, without loss of generality, that $\mathbf{M}$ is symmetric.

We are now in a position to introduce the important concept of positive definite matrices.

Definition 3.11 *A square $n \times n$ matrix* **M** *is positive definite (p.d.) if*

$$\mathbf{x} \neq \mathbf{0} \quad \Rightarrow \quad \mathbf{x}^T \mathbf{M} \mathbf{x} > 0$$

In other words, a matrix **M** is positive definite if the quadratic function $\mathbf{x}^T\mathbf{M}\mathbf{x}$ is a positive definite function. This definition implies that to every positive definite matrix is associated a positive definite function. Obviously, the converse is not true.

Geometrically, the definition of positive-definiteness can be interpreted as simply saying that the angle between a vector **x** and its image **Mx** is always less than 90^o (Figure 3.18).

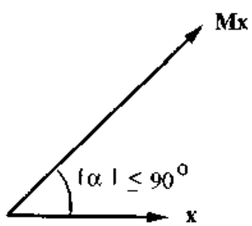


Figure 3.18 : Geometric interpretation of the positive-definiteness of a matrix M

A *necessary* condition for a square matrix **M** to be *p.d.* is that its diagonal elements be strictly positive, as can be seen by applying the above definition to the basis vectors. A famous matrix algebra result called Sylvester's theorem shows that, *assuming that* **M** *is symmetric*, a *necessary and sufficient* condition for **M** to be *p.d.* is that its principal minors (*i.e.*, M_{11} , $M_{11}M_{22} - M_{21}\,M_{12}$, ... , $\det \mathbf{M}$) all be strictly positive; or, equivalently, that all its eigenvalues be strictly positive. In particular, a symmetric *p.d.* matrix is always invertible, because the above implies that its determinant is non-zero.

A positive definite matrix **M** can always be decomposed as

$$\mathbf{M} = \mathbf{U}^T \Lambda\, \mathbf{U} \tag{3.17}$$

where **U** is a matrix of eigenvectors and satisfies $\mathbf{U}^T\mathbf{U} = \mathbf{I}$, and Λ is a diagonal matrix containing the eigenvalues of the matrix **M**. Let $\lambda_{min}(\mathbf{M})$ denote the smallest eigenvalue of **M** and $\lambda_{max}(\mathbf{M})$ the largest. Then, it follows from (3.17) that

$$\lambda_{min}(\mathbf{M})\, \|\mathbf{x}\|^2 \leq \mathbf{x}^T\mathbf{M}\mathbf{x} \leq \lambda_{max}(\mathbf{M})\|\mathbf{x}\|^2$$

This is due to the following three facts:

- $\mathbf{x}^T\mathbf{M}\,\mathbf{x} = \mathbf{x}^T\mathbf{U}^T\boldsymbol{\Lambda}\,\mathbf{U}\mathbf{x} = \mathbf{z}^T\boldsymbol{\Lambda}\mathbf{z}$, where $\mathbf{U}\,\mathbf{x} = \mathbf{z}$
- $\lambda_{min}(\mathbf{M})\,\mathbf{I} \;\leq\; \boldsymbol{\Lambda} \;\leq\; \lambda_{max}(\mathbf{M})\,\mathbf{I}$
- $\mathbf{z}^T\mathbf{z} = \|x\|^2$

The concepts of positive semi-definite, negative definite, and negative semi-definite can be defined similarly. For instance, a square $n \times n$ matrix $\mathbf{M}$ is said to be *positive semi-definite* (*p.s.d.*) if

$$\forall\, \mathbf{x}\,,\, \mathbf{x}^T\,\mathbf{M}\,\mathbf{x} \geq 0$$

By continuity, necessary and sufficient conditions for positive semi-definiteness are obtained by substituting "positive or zero" to "strictly positive" in the above conditions for positive definiteness. Similarly, a *p.s.d.* matrix is invertible only if it is actually *p.d.* Examples of *p.s.d.* matrices are $n \times n$ matrices of the form $\mathbf{M} = \mathbf{N}^T\mathbf{N}$ where $\mathbf{N}$ is a $m \times n$ matrix. Indeed,

$$\forall\, \mathbf{x}\,,\, \mathbf{x}^T\,\mathbf{N}^T\mathbf{N}\,\mathbf{x} = (\mathbf{N}\mathbf{x})^T\,(\mathbf{N}\mathbf{x}) \geq 0$$

A matrix inequality of the form

$$\mathbf{M}_1 > \mathbf{M}_2$$

(where $\mathbf{M}_1$ and $\mathbf{M}_2$ are square matrices of the same dimension) means that

$$\mathbf{M}_1 - \mathbf{M}_2 \;>\; \mathbf{0}$$

i.e., that the matrix $\mathbf{M}_1 - \mathbf{M}_2$ is positive definite. Similar notations apply to the concepts of positive semi-definiteness, negative definiteness, and negative semi-definiteness.

A time-varying matrix $\mathbf{M}(t)$ is *uniformly* positive definite if

$$\exists\, \alpha > 0,\ \forall\, t \geq 0,\ \mathbf{M}(t) \geq \alpha\,\mathbf{I}$$

A similar definition applies for uniform negative-definiteness of a time-varying matrix.

LYAPUNOV FUNCTIONS FOR LINEAR TIME-INVARIANT SYSTEMS

Given a linear system of the form $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$, let us consider a quadratic Lyapunov function candidate

$$V = \mathbf{x}^T\,\mathbf{P}\,\mathbf{x}$$

where **P** is a given symmetric positive definite matrix. Differentiating the positive definite function V along the system trajectory yields another quadratic form

$$\dot{V} = \dot{\mathbf{x}}^T \mathbf{P}\,\mathbf{x} + \mathbf{x}^T \mathbf{P}\,\dot{\mathbf{x}} = -\ \mathbf{x}^T \mathbf{Q}\,\mathbf{x} \tag{3.18}$$

where

$$\mathbf{A}^T \mathbf{P} + \mathbf{P}\,\mathbf{A} = -\,\mathbf{Q} \tag{3.19}$$

The question, thus, is to determine whether the symmetric matrix **Q** defined by the so-called *Lyapunov equation* (3.19) above, is itself *p.d.* If this is the case, then V satisfies the conditions of the basic theorem of section 3.4, and the origin is globally asymptotically stable. However, this "natural" approach may lead to inconclusive result, *i.e.*, **Q** may be not positive definite even for stable systems.

Example 3.17: Consider a second-order linear system whose **A** matrix is

$$A = \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}$$

If we take $\mathbf{P} = \mathbf{I}$, then

$$-Q = \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} = \begin{bmatrix} 0 & -4 \\ -4 & -24 \end{bmatrix}$$

The matrix **Q** is not positive definite. Therefore, no conclusion can be drawn from the Lyapunov function on whether the system is stable or not. □

A more useful way of studying a given linear system using scalar quadratic functions is, instead, to derive a positive definite matrix **P** from a given positive definite matrix **Q**, *i.e.*,

- choose a positive definite matrix **Q**
- *solve* for **P** from the Lyapunov equation (3.19)
- check whether **P** is *p.d*

If **P** is *p.d.*, then $\mathbf{x}^T\mathbf{P}\mathbf{x}$ is a Lyapunov function for the linear system and global asymptotical stability is guaranteed. Unlike the previous approach of going from a given **P** to a matrix **Q**, this technique of going from a given **Q** to a matrix **P** always leads to conclusive results for stable linear systems, as seen from the following theorem.

Theorem 3.6 *A necessary and sufficient condition for a LTI system* $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$ *to be strictly stable is that, for any symmetric p.d. matrix* $\mathbf{Q}$*, the unique matrix* $\mathbf{P}$ *solution of the Lyapunov equation (3.19) be symmetric positive definite.*

Proof: The above discussion shows that the condition is sufficient, thus we only need to show that it is also necessary. We first show that given any symmetric *p.d.* matrix $\mathbf{Q}$, there exists a symmetric *p.d.* matrix $\mathbf{P}$ verifying (3.19). We then show that for a given $\mathbf{Q}$, the matrix $\mathbf{P}$ is actually unique.

Let $\mathbf{Q}$ be a given symmetric positive definite matrix, and let

$$\mathbf{P} = \int_0^\infty \exp(\mathbf{A}^T t)\,\mathbf{Q}\,\exp(\mathbf{A}\,t)\,dt \tag{3.20}$$

One can easily show that this integral exists if and only if $\mathbf{A}$ is strictly stable. Also note that the matrix $\mathbf{P}$ thus defined is symmetric and positive definite, since $\mathbf{Q}$ is. Furthermore, we have

$$\begin{aligned}
-\mathbf{Q} &= \int_{t=0}^{\infty} d[\,\exp(\mathbf{A}^T t)\,\mathbf{Q}\,\exp(\mathbf{A}\,t)\,] \\
&= \int_{t=0}^{\infty} [\,\mathbf{A}^T \exp(\mathbf{A}^T t)\,\mathbf{Q}\,\exp(\mathbf{A}\,t) + \exp(\mathbf{A}^T t)\,\mathbf{Q}\,\exp(\mathbf{A}\,t)\,\mathbf{A}\,]\,dt \\
&= \mathbf{A}^T\mathbf{P} + \mathbf{P}\,\mathbf{A}
\end{aligned}$$

where the first equality comes from the stability of $\mathbf{A}$ (which implies that $\exp(\mathbf{A}\infty) = \mathbf{0}$), the second from differentiating the exponentials explicitly, and the third from the fact that $\mathbf{A}$ is constant and therefore can be taken out of the integrals.

The uniqueness of $\mathbf{P}$ can be verified similarly by noting that another solution $\mathbf{P}_1$ of the Lyapunov equation would necessarily verify

$$\begin{aligned}
\mathbf{P}_1 &= -\int_{t=0}^{\infty} d[\,\exp(\mathbf{A}^T t)\,\mathbf{P}_1\,\exp(\mathbf{A}\,t)\,] \\
&= -\int_{t=0}^{\infty} \exp(\mathbf{A}^T t)\,(\,\mathbf{A}^T\mathbf{P}_1 + \mathbf{P}_1\,\mathbf{A}\,)\,\exp(\mathbf{A}\,t)\,]\,dt \\
&= \int_0^{\infty} \exp(\mathbf{A}^T t)\,\mathbf{Q}\,\exp(\mathbf{A}\,t)\,dt \;=\; \mathbf{P}
\end{aligned}$$

An alternate proof of uniqueness is the elegant original proof given by Lyapunov, which makes direct use of fundamental algebra results. The Lyapunov equation (3.19) can be interpreted as defining a linear map from the n^2 components of the matrix **P** to the n^2 components of the matrix **Q**, where **P** and **Q** are arbitrary (not necessarily symmetric *p.d.*) square matrices. Since (3.20) actually shows the existence of a solution **P** for *any* square matrix **Q**, the range of this linear map is full, and therefore its null-space is reduced to **0**. Thus, for any **Q**, the solution **P** is unique. □

The above theorem shows that *any* positive definite matrix **Q** can be used to determine the stability of a linear system. A simple choice of **Q** is the identity matrix.

Example 3.18: Consider again the second-order system of Example 3.17. Let us take $\mathbf{Q} = \mathbf{I}$ and denote **P** by

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

where, due to the symmetry of **P**, $p_{21} = p_{12}$. Then the Lyapunov equation is

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix} + \begin{bmatrix} 0 & -8 \\ 4 & -12 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

whose solution is

$$p_{11} = 5/16\,, \quad p_{12} = p_{22} = 1/16$$

The corresponding matrix

$$\mathbf{P} = \frac{1}{16} \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$$

is positive definite, and therefore the linear system is globally asymptotically stable. Note that we have solved for **P** directly, without using the more cumbersome expression (3.20). □

Even though the choice $\mathbf{Q} = \mathbf{I}$ is motivated by computational simplicity, it has a surprising property: the resulting Lyapunov analysis allows us to get the best estimate of the state convergence rate, as we shall see in section 3.5.5.

3.5.2 Krasovskii's Method

Let us now come back to the problem of finding Lyapunov functions for general nonlinear systems. Krasovskii's method suggests a simple form of Lyapunov function candidate for autonomous nonlinear systems of the form (3.2), namely, $V = \mathbf{f}^T\mathbf{f}$. The basic idea of the method is simply to check whether this particular choice indeed leads to a Lyapunov function.

Theorem 3.7 (Krasovskii) *Consider the autonomous system defined by (3.2), with the equilibrium point of interest being the origin. Let* $\mathbf{A}(\mathbf{x})$ *denote the Jacobian matrix of the system, i.e.,*

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$$

If the matrix $\mathbf{F} = \mathbf{A} + \mathbf{A}^T$ *is negative definite in a neighborhood* Ω, *then the equilibrium point at the origin is asymptotically stable. A Lyapunov function for this system is*

$$V(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})\,\mathbf{f}(\mathbf{x})$$

If Ω *is the entire state space and, in addition,* $V(\mathbf{x}) \to \infty$ *as* $\|\mathbf{x}\| \to \infty$, *then the equilibrium point is globally asymptotically stable.*

Proof: First, let us prove that the negative definiteness of $\mathbf{F}$ implies that $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$ for $\mathbf{x} \neq \mathbf{0}$.

Since the square matrix $\mathbf{F}(\mathbf{x})$ is negative definite for non-zero $\mathbf{x}$, one can show that the Jacobian matrix $\mathbf{A}(\mathbf{x})$ is invertible, by contradiction. Indeed, assume that $\mathbf{A}$ is singular. Then one can find a non-zero vector $\mathbf{y}_o$ such that $\mathbf{A}(\mathbf{x})\,\mathbf{y}_o = \mathbf{0}$. Since

$$\mathbf{y}_o^T\mathbf{F}\,\mathbf{y}_o = 2\,\mathbf{y}_o^T\mathbf{A}\,\mathbf{y}_o$$

the singularity of $\mathbf{A}$ implies that $\mathbf{y}_o^T\mathbf{A}\,\mathbf{y}_o = 0$, which contradicts the assumed negative definiteness of $\mathbf{F}$.

The invertibility and continuity of $\mathbf{A}$ guarantee that the function $\mathbf{f}(\mathbf{x})$ can be *uniquely* inverted. This implies that the dynamic system (3.2) has only one equilibrium point in Ω (otherwise different equilibrium points would correspond to the same value of $\mathbf{f}$), *i.e.*, that $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$ for $\mathbf{x} \neq \mathbf{0}$.

We can now show the asymptotic stability of the origin. Given the above result, the scalar function $V(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})\,\mathbf{f}(\mathbf{x})$ is positive definite. Using the fact that $\dot{\mathbf{f}} = \mathbf{A}\,\mathbf{f}$, the derivative of V can be written

$$\dot{V}(\mathbf{x}) = \mathbf{f}^T\dot{\mathbf{f}} + \dot{\mathbf{f}}^T\mathbf{f} = \mathbf{f}^T\mathbf{A}\,\mathbf{f} + \mathbf{f}^T\mathbf{A}^T\mathbf{f} = \mathbf{f}^T\mathbf{F}\,\mathbf{f}$$

The negative definiteness of $\mathbf{F}$ implies the negative definiteness of $\dot{V}$. Therefore, according to Lyapunov's direct method, the equilibrium state at the origin is asymptotically stable. The global asymptotic stability of the system is guaranteed by the global version of Lyapunov's direct method. □

Let us illustrate the use of Krasovskii's theorem on a simple example.

Example 3.19: Consider the nonlinear system

$$\dot{x}_1 = -6x_1 + 2x_2$$

$$\dot{x}_2 = 2x_1 - 6x_2 - 2x_2^3$$

We have

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} -6 & 2 \\ 2 & -6-6x_2^2 \end{bmatrix} \qquad \mathbf{F} = \mathbf{A} + \mathbf{A}^T = \begin{bmatrix} -12 & 4 \\ 4 & -12-12x_2^2 \end{bmatrix}$$

The matrix **F** is easily shown to be negative definite over the whole state space. Therefore, the origin is asymptotically stable, and a Lyapunov function candidate is

$$V(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})\,\mathbf{f}(\mathbf{x}) = (-6x_1 + 2x_2)^2 + (2x_1 - 6x_2 - 2x_2^3)^2$$

Since $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$, the equilibrium state at the origin is globally asymptotically stable. □

While the use of the above theorem is very straightforward, its applicability is limited in practice, because the Jacobians of many systems do not satisfy the negative definiteness requirement. In addition, for systems of high order, it is difficult to check the negative definiteness of the matrix **F** for *all* **x**.

An immediate generalization of Krasovskii's theorem is as follows:

Theorem 3.8 (Generalized Krasovskii Theorem) *Consider the autonomous system defined by (3.2), with the equilibrium point of interest being the origin, and let* $\mathbf{A}(\mathbf{x})$ *denote the Jacobian matrix of the system. Then, a sufficient condition for the origin to be asymptotically stable is that there exist two symmetric positive definite matrices* **P** *and* **Q**, *such that* $\forall \mathbf{x} \neq \mathbf{0}$, *the matrix*

$$\mathbf{F}(\mathbf{x}) = \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q}$$

is negative semi-definite in some neighborhood Ω *of the origin. The function* $V(\mathbf{x}) = \mathbf{f}^T\mathbf{P}\mathbf{f}$ *is then a Lyapunov function for the system. If the region* Ω *is the whole state space, and if in addition,* $V(\mathbf{x}) \to \infty$ *as* $\|\mathbf{x}\| \to \infty$, *then the system is globally asymptotically stable.*

Proof: This theorem can be proven similarly. The positive definiteness of $V(\mathbf{x})$ can be derived as before. Furthermore, the derivative of V can be computed as

$$\dot{V} = \frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) = \mathbf{f}^T \mathbf{P} \mathbf{A}(\mathbf{x}) \mathbf{f} + \mathbf{f}^T \mathbf{P} \mathbf{A}^T(\mathbf{x}) \mathbf{P} \mathbf{f} = \mathbf{f}^T \mathbf{F} \mathbf{f} - \mathbf{f}^T \mathbf{Q} \mathbf{f}$$

Because $\mathbf{F}$ is negative semi-definite and $\mathbf{Q}$ is positive definite, $\dot{V}$ is negative definite and the equilibrium point at the origin is asymptotically stable. If $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$, the global version of Lyapunov's direct method indicates the global asymptotic stability of the system. □

3.5.3 The Variable Gradient Method

The variable gradient method is a formal approach to constructing Lyapunov functions. It involves assuming a certain form for the gradient of an unknown Lyapunov function, and then finding the Lyapunov function itself by integrating the assumed gradient. For low order systems, this approach sometimes leads to the successful discovery of a Lyapunov function.

To start with, let us note that a scalar function $V(\mathbf{x})$ is related to its gradient ∇V by the integral relation

$$V(\mathbf{x}) = \int_0^{\mathbf{x}} \nabla V \, d\mathbf{x}$$

where $\nabla V = \{\partial V/\partial x_1, \ldots\ldots, \partial V/\partial x_n\}^T$. In order to recover a unique scalar function V from the gradient ∇V, the gradient function has to satisfy the so-called curl conditions

$$\frac{\partial \nabla V_i}{\partial x_j} = \frac{\partial \nabla V_j}{\partial x_i} \qquad (i, j = 1, 2, \ldots, n)$$

Note that the i^{th} component ∇V_i is simply the directional derivative $\partial V/\partial x_i$. For instance, in the case $n = 2$, the above simply means that

$$\frac{\partial \nabla V_1}{\partial x_2} = \frac{\partial \nabla V_2}{\partial x_1}$$

The principle of the variable gradient method is to assume a specific form *for the gradient* ∇V, instead of assuming a specific form for the Lyapunov function V itself. A simple way is to assume that the gradient function is of the form

$$\nabla V_i = \sum_{j=1}^{n} a_{ij} x_j \tag{3.21}$$

where the a_{ij}'s are coefficients to be determined. This leads to the following procedure for seeking a Lyapunov function V:

- assume that ∇V is given by (3.21) (or another form)
- solve for the coefficients a_{ij} so as to satisfy the curl equations
- restrict the coefficients in (3.21) so that $\dot{V}$ is negative semi-definite (at least locally)
- compute V from ∇V by integration;
- check whether V is positive definite

Since satisfaction of the curl conditions implies that the above integration result is independent of the integration path, it is usually convenient to obtain V by integrating along a path which is parallel to each axis in turn, *i.e.*,

$$V(\mathbf{x}) = \int_o^{x_1} \nabla V_1(x_1,0,\ldots,0)\,dx_1 + \int_o^{x_2} \nabla V_2(x_1,x_2,0,\ldots,0)dx_2 + \ \ldots$$

$$+ \int_o^{x_n} \nabla V_n(x_1,x_2,\ldots,x_n)\,dx_n$$

Example 3.20: Let us use the variable gradient method to find a Lyapunov function for the nonlinear system

$$\dot{x}_1 = -2x_1$$

$$\dot{x}_2 = -2x_2 + 2x_1x_2^{\,2}$$

We assume that the gradient of the undetermined Lyapunov function has the following form

$$\nabla V_1 = a_{11}x_1 + a_{12}x_2$$

$$\nabla V_2 = a_{21}x_1 + a_{22}x_2$$

The curl equation is

$$\frac{\partial \nabla V_1}{\partial x_2} = \frac{\partial \nabla V_2}{\partial x_1}$$

$$a_{12} + x_2 \frac{\partial a_{12}}{\partial x_2} = a_{21} + x_1 \frac{\partial a_{21}}{\partial x_1}$$

If the coefficients are chosen to be

$$a_{11} = a_{22} = 1, a_{12} = a_{21} = 0$$

which leads to

$$\nabla V_1 = x_1 \qquad \nabla V_2 = x_2$$

then $\dot{V}$ can be computed as

$$\dot{V} = \nabla V \, \dot{\mathbf{x}} = -2x_1^2 - 2x_2^2(1 - x_1x_2)$$

Thus, $\dot{V}$ is locally negative definite in the region $(1 - x_1x_2) > 0$. The function V can be computed as

$$V(\mathbf{x}) = \int_o^{x_1} x_1 dx_1 + \int_o^{x_2} x_2 dx_2 = \frac{x_1^2 + x_2^2}{2} \tag{3.22}$$

This is indeed positive definite, and therefore the asymptotic stability is guaranteed.

Note that (3.22) is not the only Lyapunov function obtainable by the variable gradient method. For example, by taking

$$a_{11} = 1, \; a_{12} = x_2^2$$

$$a_{21} = 3\,x_2^2 \;, \; a_{22} = 3$$

we obtain the positive definite function

$$V = \frac{x_1^2}{2} + \frac{3}{2}x_2^2 + x_1x_2^3 \tag{3.23}$$

whose derivative is

$$\dot{V} = -2x_1^2 - 6x_2^2 - 2x_2^2(x_1x_2 - 3x_1^2x_2^2)$$

One easily verifies that $\dot{V}$ is a locally negative definite function (noting that the quadratic terms are dominant near the origin), and therefore, (3.23) represents another Lyapunov function for the system. □

3.5.4 Physically Motivated Lyapunov Functions

The Lyapunov functions in the above sections 3.5.1-3.5.3, and in a number of examples earlier in section 3.4, have been obtained from a *mathematical* point of view, *i.e.*, we examined the mathematical features of the given differential equations and searched for Lyapunov function candidates V that can make $\dot{V}$ negative. We did not pay much attention to where the dynamic equations came from and what properties the physical systems had. However, this purely mathematical approach, though effective for simple systems, is often of little use for complicated dynamic equations. On the other hand, if engineering insight and physical properties are properly exploited, an elegant and powerful Lyapunov analysis may be possible for very complex systems.

Example 3.21: Global asymptotic stability of a robot position controller

A fundamental task in robotic applications is for robot manipulators to transfer objects from one point to another, the so-called robot position control problem. In the last decade, engineers had been routinely using P.D. (proportional plus derivative) controllers to control robot arms. However, there was no theoretical justification for the stability of such control systems, because the dynamics of a robot is highly nonlinear.

A robot arm consists a number of links connected by rotational or translational joints, with the last link equipped with some end-effector (Figure 3.19). The dynamics of an n-link robot arm can be expressed by a set of n equations,

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} \tag{3.24}$$

where $\mathbf{q}$ is an n-dimensional vector describing the joint positions of the robot, $\boldsymbol{\tau}$ is the vector of input torques, $\mathbf{g}$ is the vector of gravitational torques, $\mathbf{b}$ represents the Coriolis and centripetal forces caused by the motion of the links, and $\mathbf{H}$ the $n{\times}n$ inertia matrix of the robot arm. Consider a controller simply composed of a P.D. term and a gravity compensation term

$$\boldsymbol{\tau} = -\mathbf{K}_D\,\dot{\mathbf{q}} - \mathbf{K}_P\,\mathbf{q} + \mathbf{g}(\mathbf{q}) \tag{3.25}$$

where $\mathbf{K}_D$ and $\mathbf{K}_P$ are constant positive definite $n{\times}n$ matrices. It is almost impossible to use trial-and error to search for a Lyapunov function for the closed loop dynamics defined by (3.24) and (3.25), because (3.24) contains hundreds of terms for the 5-link or 6-link robot arms commonly found in industry. Therefore, it seems very difficult to show that $\dot{\mathbf{q}} \to \mathbf{0}$ and $\mathbf{q} \to \mathbf{0}$.

With the aid of physical insights, however, a Lyapunov function can be successfully found for such complex robotic systems. First, note that the inertia matrix $\mathbf{H}(\mathbf{q})$ is positive definite for any $\mathbf{q}$. Second, the P.D. control term can be interpreted as mimicking a combination of dampers and springs. This suggests the following Lyapunov function candidate

$$V = \frac{1}{2}\left[\dot{\mathbf{q}}^T\mathbf{H}\,\dot{\mathbf{q}} + \mathbf{q}^T\mathbf{K}_p\,\mathbf{q}\right]$$

where the first term represents the kinetic energy of the manipulator, and the second term denotes the "artificial potential energy" associated with the virtual spring in the control law (3.25).

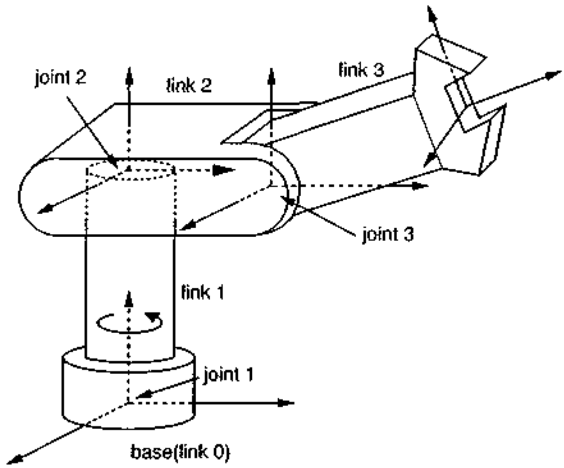


Figure 3.19 : A robot manipulator

In computing the derivative of this function, one can use the energy theorem in mechanics, which states that the rate of change of kinetic energy in a mechanical system is equal to the power provided by the external forces. Therefore,

$$\dot{V} = \dot{\mathbf{q}}^T (\boldsymbol{\tau} - \mathbf{g}) + \dot{\mathbf{q}}^T \mathbf{K}_P \mathbf{q}$$

Substitution of the control law (3.25) in the above equation then leads to

$$\dot{V} = -\dot{\mathbf{q}}^T \mathbf{K}_D \dot{\mathbf{q}}$$

Since the arm cannot get "stuck" at any position such that $\mathbf{q} \neq \mathbf{0}$ (which can be easily shown by noting that acceleration is non-zero in such situations), the robot arm must settle down at $\dot{\mathbf{q}} = \mathbf{0}$ and $\mathbf{q} = \mathbf{0}$, according to the invariant set theorem. Thus, the system is actually *globally* asymptotically stable. □

Two lessons can be learned from this practical example. The first is that one should use as many physical properties as possible in analyzing the behavior of a system. The second lesson is that physical concepts like energy may lead us to some uniquely powerful choices of Lyapunov functions. Physics will play a major role in the development of the multi-input nonlinear controllers of chapter 9.

3.5.5 Performance Analysis

In the preceding sections, we have been primarily concerned with using Lyapunov functions for stability analysis. But sometimes Lyapunov functions can further provide estimates of the transient performance of stable systems. In particular, they can allow us to estimate the convergence rate of asymptotically stable linear or nonlinear systems. In this section, we first derive a simple lemma on differential inequalities. We then show how Lyapunov analysis may be used to determine the convergence rates of linear and nonlinear systems.

A SIMPLE CONVERGENCE LEMMA

Lemma: *If a real function $W(t)$ satisfies the inequality*

$$\dot{W}(t) + \alpha W(t) \leq 0 \tag{3.26}$$

where α is a real number. Then

$$W(t) \leq W(0)\, e^{-\alpha t}$$

Proof: Let us define a function $Z(t)$ by

$$Z(t) = \dot{W} + \alpha W \tag{3.27}$$

Equation (3.26) implies that $Z(t)$ is non-positive. The solution of the first-order equation (3.27) is

$$W(t) = W(0)\, e^{-\alpha t} + \int_0^t e^{-\alpha(t-r)} Z(t)\, dr$$

Because the second term in the right-hand-side of the above equation is non-positive, one has

$$W(t) \leq W(0)\, e^{-\alpha t} \qquad \square$$

The above lemma implies that, if W is a non-negative function, the satisfaction of (3.26) guarantees the exponential convergence of W to zero. In using Lyapunov's direct method for stability analysis, it is sometimes possible to manipulate $\dot{V}$ into the form of (3.26). In such a case, the exponential convergence of V and the convergence rate can be inferred and, in turn, the exponential convergence rate of the state may then be determined. In later chapters, we will provide examples of using this lemma for estimating convergence rates of nonlinear or adaptive control systems.

ESTIMATING CONVERGENCE RATES FOR LINEAR SYSTEMS

Now let us evaluate the convergence rate of a stable linear system based on the Lyapunov analysis described in section 3.5.1. Let us denote the largest eigenvalue of the matrix $\mathbf{P}$ by $\lambda_{\max}(\mathbf{P})$, the smallest eigenvalue of $\mathbf{Q}$ by $\lambda_{\min}(\mathbf{Q})$, and their ratio $\lambda_{min}(\mathbf{Q})/\lambda_{max}(\mathbf{P})$ by γ. The positive definiteness of $\mathbf{P}$ and $\mathbf{Q}$ implies that these scalars are all strictly positive. Since matrix theory shows that

$$\mathbf{P} \leq \lambda_{max}(\mathbf{P})\,\mathbf{I} \qquad\qquad \lambda_{min}(\mathbf{Q})\,\mathbf{I} \leq \mathbf{Q}$$

we have

$$\mathbf{x}^T\mathbf{Q}\,\mathbf{x} \;\geq\; \frac{\lambda_{min}(\mathbf{Q})}{\lambda_{max}(\mathbf{P})}\,\mathbf{x}^T[\lambda_{max}(\mathbf{P})\mathbf{I}]\mathbf{x} \;\geq\; \gamma V$$

This and (3.18) imply that

$$\dot{V} \leq -\gamma V$$

This, according to lemma, means that

$$\mathbf{x}^T\mathbf{P}\,\mathbf{x} \leq V(0)\,e^{-\gamma t}$$

This, together with the fact $\mathbf{x}^T\mathbf{P}\,\mathbf{x} \geq \lambda_{min}(\mathbf{P})\|\mathbf{x}(t)\|^2$, implies that the state $\mathbf{x}$ converges to the origin with a rate of at least $\gamma/2$.

One might naturally wonder how this convergence rate estimate varies with the choice of $\mathbf{Q}$, and how it relates to the familiar notion of dominant pole in linear theory. An interesting result is that *the convergence rate estimate is largest for* $\mathbf{Q} = \mathbf{I}$. Indeed, let $\mathbf{P}_o$ be the solution of the Lyapunov equation corresponding to $\mathbf{Q} = \mathbf{I}$:

$$\mathbf{A}^T\,\mathbf{P}_o + \mathbf{P}_o\,\mathbf{A} = -\,\mathbf{I}$$

and let $\mathbf{P}$ be the solution corresponding to some other choice of $\mathbf{Q}$

$$\mathbf{A}^T\,\mathbf{P} + \mathbf{P}\,\mathbf{A} = -\,\mathbf{Q}_1$$

Without loss of generality, we can assume that $\lambda_{min}(\mathbf{Q}_1) = 1$, since rescaling $\mathbf{Q}_1$ will rescale $\mathbf{P}$ by the same factor, and therefore will not affect the value of the corresponding γ. Subtracting the above two equations yields

$$\mathbf{A}^T\,(\mathbf{P} - \mathbf{P}_o) + (\mathbf{P} - \mathbf{P}_o)\,\mathbf{A} = -\,(\mathbf{Q}_1 - \mathbf{I})$$

Now since $\lambda_{min}(\mathbf{Q}_1) = 1 = \lambda_{max}(\mathbf{I})$, the matrix $(\mathbf{Q}_1 - \mathbf{I})$ is positive semi-definite, and hence the above equation implies that $(\mathbf{P} - \mathbf{P}_o)$ is positive semi-definite. Therefore

$$\lambda_{max}(\mathbf{P}) \geq \lambda_{max}(\mathbf{P}_o)$$

Since $\lambda_{min}(\mathbf{Q}_1) = 1 = \lambda_{min}(\mathbf{I})$, the convergence rate estimate

$$\gamma = \lambda_{min}(\mathbf{Q})/\lambda_{max}(\mathbf{P})$$

corresponding to $\mathbf{Q} = \mathbf{I}$ is larger than (or equal to) that corresponding to $\mathbf{Q} = \mathbf{Q}_1$.

If the stable matrix $\mathbf{A}$ is symmetric, then the meaning of this "optimal" value of γ, corresponding to the choice $\mathbf{Q} = \mathbf{I}$, can be interpreted easily. Indeed, all eigenvalues of $\mathbf{A}$ are then real, and furthermore $\mathbf{A}$ is diagonalizable, *i.e.*, there exists a change of state coordinates such that in these coordinates $\mathbf{A}$ is diagonal. One immediately verifies that, in these coordinates, the matrix $\mathbf{P} = -1/2\ \mathbf{A}^{-1}$ verifies the Lyapunov equation for $\mathbf{Q} = \mathbf{I}$, and that therefore the corresponding $\gamma/2$ is simply the absolute value of the *dominant pole* of the linear system. Furthermore, γ is obviously independent of the choice of state coordinates.

ESTIMATING CONVERGENCE RATES FOR NONLINEAR SYSTEMS

The estimation of convergence rate for nonlinear systems also involves manipulating the expression of $\dot{V}$ so as to obtain an explicit estimate of V. The difference lies in that, for nonlinear systems, V and $\dot{V}$ are not necessarily quadratic functions of the states.

Example 3.22: Consider again the system in Example 3.8. Given the chosen Lyapunov function candidate $V = \|\mathbf{x}\|^2$, the derivative $\dot{V}$, can be written

$$\dot{V} = 2\,V(V-1)$$

that is,

$$\frac{dV}{V(1-V)} = -2\,dt$$

The solution of this equation is easily found to be

$$V(\mathbf{x}) = \frac{\alpha e^{-2t}}{1 + \alpha e^{-2t}}$$

where

$$\alpha = \frac{V(0)}{1 - V(0)}$$

If $\|\mathbf{x}(0)\|^2 = V(0) < 1$, *i.e.*, if the trajectory starts inside the unit circle, then $\alpha > 0$, and

$$V(t) < \alpha e^{-2t}$$

This implies that the norm $\|\mathbf{x}(t)\|$ of the state vector converges to zero exponentially, with a rate of 1.

However, if the trajectory starts outside the unit circle, *i.e.*, if $V(0) > 1$, then $\alpha < 0$, so that

$V(t)$ and therefore $\|x\|$ tend to infinity in a finite time (the system is said to exhibit finite escape time, or "explosion"). □

3.6 Control Design Based on Lyapunov's Direct Method

The previous sections have been concerned with using Lyapunov's direct method for system analysis. In doing the analysis, we have implicitly presumed that certain control laws have been chosen for the systems. However, in many control problems, the task is to find an appropriate control law for a given plant. In the following, we briefly discuss how to apply Lyapunov's direct method for designing stable control systems. Many of the controller design methods we will describe in chapters 6-9 are actually based on Lyapunov concepts.

There are basically two ways of using Lyapunov's direct method for control design, and both have a trial and error flavor. The first technique involves hypothesizing one form of control law and then finding a Lyapunov function to justify the choice. The second technique, conversely, requires hypothesizing a Lyapunov function candidate and then finding a control law to make this candidate a real Lyapunov function.

We saw an application of the first technique in the robotic P.D. control example in section 3.5.4, where a P.D. controller is chosen based on physical intuition, and a Lyapunov function is found to show the global asymptotic convergence of the resulting closed-loop system. The second technique can be illustrated on the following simple example.

Example 3.23: Regulator Design

Consider the problem of stabilizing the system

$$\ddot{x} - \dot{x}^3 + x^2 = u$$

i.e., to bring it to equilibrium at $x \equiv 0$. Based on Example 3.14, it is sufficient to choose a continuous control law u of the form

$$u = u_1(\dot{x}) + u_2(x)$$

where

$$\dot{x}(\dot{x}^3 + u_1(\dot{x})) < 0 \quad \text{for} \quad \dot{x} \neq 0$$

$$x(x^2 - u_2(x)) > 0 \quad \text{for} \quad x \neq 0$$

The above inequalities also imply that globally stabilizing controllers can be designed even in the presence of some uncertainty on the dynamics. For instance, the system

$$\ddot{x} + \alpha_1 \dot{x}^3 + \alpha_2 x^2 = u$$

where the constants α_1 and α_2 are unknown, but such that $\alpha_1 > -2$ and $|\alpha_2| < 5$, can be globally stabilized using the control law

$$u = -2\dot{x}^3 - 5x|x| \qquad \square$$

For some classes of nonlinear systems, systematic design procedures have been developed based on these above two techniques, as can be seen in chapter 7 on the sliding control design methodology, in chapter 8 on adaptive control, and in chapter 9 on physically-motivated designs.

Finally, it is important to notice that, just as a nonlinear system may be globally asymptotically stable while its linear approximation is only marginally stable, a nonlinear system may be controllable while its linear approximation is not. Consider for instance the system

$$\ddot{x} + \dot{x}^5 = x^2 u$$

This system can be made to converge asymptotically to zero simply by letting $u = -x$. However, its linear approximation at $x = 0$ and $u = 0$ is $\ddot{x} \approx 0$, and therefore is uncontrollable!

3.7 Summary

Stability is a fundamental issue in control system analysis and design. Various concepts of stability, such as Lyapunov stability, asymptotic stability, exponential stability, and global asymptotic or exponential stability, must be defined in order to accurately characterize the complex and rich stability behaviors exhibited by nonlinear systems.

Since analytical solutions of nonlinear differential equations usually cannot be obtained, the two methods of Lyapunov are of major importance in the determination of nonlinear system stability.

The linearization method is concerned with the small motion of nonlinear systems around equilibrium points. It is mainly of theoretical value, justifying the use of linear control for the design and analysis of weakly nonlinear physical systems.

The direct method, based on so-called Lyapunov functions, is not restricted to small motions. In principle, it is applicable to essentially all dynamic systems, whether linear or nonlinear, continuous-time or discrete-time , of finite or infinite order, and in small or large motion. However, the method suffers from the common difficulty of finding a Lyapunov function for a given system. Since there is no generally effective approach for finding Lyapunov functions, one has to use trial-and-error, experience, and intuition to search for appropriate Lyapunov functions. Physical properties (such as energy conservation) and physical insights may be exploited in formulating Lyapunov functions, and may lead to uniquely powerful choices. Simple mathematical techniques, such as, *e.g.*, Krasovskii's method or the variable gradient method, can also be of help.

Generally, the application of Lyapunov theory to *controller design* is more easily rewarding. This is because, in design, one often has the freedom to deliberately modify the dynamics (by designing the controller) in such a way that a chosen scalar function becomes a Lyapunov function for the closed-loop system. In the second part of the book, we will see many applications of Lyapunov theory to the construction of effective nonlinear control systems.

3.8 Notes and References

In Lyapunov's original work [Lyapunov, 1892], the linearization method (which, today, is sometimes incorrectly referred to as the first method) is simply given as an example of application of the direct (or second) method. The first method was the so-called method of exponents, which is used today in the analysis of chaos.

Many references can be found on Lyapunov theory, *e.g.*, [La Salle and Lefschetz, 1961] (on which the invariant set theorems are based), [Kalman and Bertram, 1960; Hahn, 1967; Yoshizawa, 1966]. An inspiring discussion of the role of scalar summarizing functions in science, along with a very readable introduction to elementary Lyapunov theory, can be found in [Luenberger, 1979].

Examples 3.1 and 3.13 are adapted from [Luenberger, 1979]. Examples 3.3 and 3.8 are adapted from [Vidyasagar, 1978]. The counterexample in Figure 3.12 is from [Hahn, 1967]. Example 3.14 is adapted from [La Salle and Lefschetz, 1961; Vidyasagar, 1978]. The variable gradient method in subsection 3.5.3 is adapted from [Ogata, 1970]. The robotic example of section 3.5.4 is based on [Takegaki and Arimoto, 1981]. The remark on the "optimal" choice of Q in section 3.5.5 is from [Vidyasagar, 1982]. A detailed study of Krasovskii's theorems can be found in [Krasovskii, 1963].

3.9 Exercises

3.1 The norm used in the definitions of stability need not be the usual Euclidian norm. If the state-space is of finite dimension n (*i.e.*, the state vector has n components), stability and its type are independent of the choice of norm (all norms are "equivalent"), although a particular choice of norm may make analysis easier. For $n = 2$, draw the unit balls corresponding to the following norms:

(i) $\|\mathbf{x}\|^2 = (x_1)^2 + (x_2)^2$ (Euclidian norm)

(ii) $\|\mathbf{x}\|^2 = (x_1)^2 + 5\,(x_2)^2$

(iii) $\|\mathbf{x}\| = |x_1| + |x_2|$

(iv) $\|\mathbf{x}\| = \mathrm{Sup}(\,|x_1|\,,\,|x_2|\,)$

Recall that a ball $\mathbf{B}(\mathbf{x}_o\,, R)$, of center $\mathbf{x}_o$ and radius R, is the set of $\mathbf{x}$ such that $\|\mathbf{x} - \mathbf{x}_o\| \leq R$, and that the unit ball is $\mathbf{B}(\mathbf{0},\, 1)$.

3.2 For the following systems, find the equilibrium points and determine their stability. Indicate whether the stability is asymptotic, and whether it is global.

(*a*) $\dot{x} = -x^3 + \sin^4 x$

(*b*) $\dot{x} = (5 - x)^5$

(*c*) $\ddot{x} + \dot{x}^5 + x^7 = x^2 \sin^8 x \cos^2 3x$

(*d*) $\ddot{x} + (x-1)^4\,\dot{x}^7 + x^5 = x^3 \sin^3 x$

(*e*) $\ddot{x} + (x-1)^2\,\dot{x}^7 + x = \sin(\pi x/2)$

3.3 For the Van der Pol oscillator of Example 3.3, demonstrate the existence of a limit cycle using the linearization method.

3.4 This exercise, adapted from [Hahn, 1967], provides an example illustrating the motivation of the radial unboundedness condition in Theorem 3.3. Consider the second-order system

$$\dot{x}_1 = -\frac{6x_1}{z^2} + 2x_2$$

$$\dot{x}_2 = -\frac{2\,(x_1 + x_2)}{z_2}$$

with $z = 1 + x_1^{\,2}$. On the hyperbola $x_2^h = 2/(x_1 - \sqrt{2}\,)$, the system trajectory slope is

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{-1}{1 + 2^{3/2}x_1 + 2x_1^{\,2}}$$

while the slope of the hyperbola is

$$\frac{dx_2^{\,h}}{dx_1} = \frac{-1}{1 - 2^{3/2}x_1 + x_1^{\,2}/2}$$

Note that for $x_1 > \sqrt{2}$, the first expression is larger than the second, implying that the trajectories cannot cut the branch of the hyperbola which lies in the first quadrant, in the direction toward the axes (since on the hyperbola we have $\dot{x}_1 > 0$ if $x_1 > \sqrt{2}$). Thus, there are trajectories which do not tend toward the origin, indicating the lack of global asymptotic stability. Use the scalar function

$$V(\mathbf{x}) = \frac{x_1^{\,2}}{2} + x_2^{\,2}$$

to analyze the stability of the above system.

3.5 Determine regions of attraction of the pendulum, using as Lyapunov functions the pendulum's total energy, and the modified Lyapunov function of page 67. Comment on the two results.

3.6 Show that given a constant matrix $\mathbf{M}$ and any time-varying vector $\mathbf{x}$, the time-derivative of the scalar $\mathbf{x}^T\mathbf{M}\,\mathbf{x}$ can be written

$$\frac{d}{dt}\mathbf{x}^T\mathbf{M}\,\mathbf{x} = \mathbf{x}^T(\mathbf{M}+\mathbf{M}^T)\,\dot{\mathbf{x}} = \dot{\mathbf{x}}^T(\mathbf{M}+\mathbf{M}^T)\,\mathbf{x}$$

and that, if $\mathbf{M}$ is symmetric, it can also be written

$$\frac{d}{dt}\mathbf{x}^T\mathbf{M}\,\mathbf{x} = 2\,\mathbf{x}^T\mathbf{M}\,\dot{\mathbf{x}} = 2\,\dot{\mathbf{x}}^T\mathbf{M}\,\mathbf{x}$$

3.7 Consider an $n \times n$ matrix $\mathbf{M}$ of the form $\mathbf{M} = \mathbf{N}^T\mathbf{N}$, where $\mathbf{N}$ is a $m \times n$ matrix. Show that $\mathbf{M}$ is *p.d.* if, and only if, $m \geq n$ and N has full rank.

3.8 Show that if $\mathbf{M}$ is a symmetric matrix such that

$$\forall\,\mathbf{x}\,,\ \mathbf{x}^T\mathbf{M}\mathbf{x} = 0$$

then $\mathbf{M} = \mathbf{0}$.

3.9 Show that if symmetric *p.d.* matrices $\mathbf{P}$ and $\mathbf{Q}$ exist such that

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\,\mathbf{A} + 2\lambda\mathbf{P} = -\mathbf{Q}$$

then all the eigenvalues of $\mathbf{A}$ have a real part strictly less than $-\lambda$. (Adapted from [Luenberger, 1979].)

3.10 Consider the system

$$\mathbf{A}_1\,\ddot{\mathbf{y}} + \mathbf{A}_2\,\dot{\mathbf{y}} + \mathbf{A}_3\,\mathbf{y} = \mathbf{0}$$

where the $2n \times 1$ vector $\mathbf{x} = [\mathbf{y}^T \;\; \dot{\mathbf{y}}^T]^T$ is the state, and the $n \times n$ matrices $\mathbf{A}_j$ are all symmetric positive definite. Show that the system is globally asymptotically stable, with $\mathbf{0}$ as a unique equilibrium point.

3.11 Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} \qquad y = \mathbf{c}^T\mathbf{x}$$

Use the invariant set theorem to show that if the system is observable, and if there exists a symmetric *p.d.* matrix $\mathbf{P}$ such that

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\,\mathbf{A} = -\mathbf{c}\mathbf{c}^T$$

then the system is asymptotically stable.

Can the result be derived using the direct method? (Adapted from [Luenberger, 1979].)

3.12 Use Krasovskii's theorem to justify Lyapunov's linearization method.

3.13 Consider the system

$$\dot{x} = 4x^2y - f_1(x)\,(x^2 + 2y^2 - 4)$$

$$\dot{y} = -2x^3 - f_2(y)\,(x^2 + 2y^2 - 4)$$

where the continuous functions f_1 and f_2 have the same sign as their argument. Show that the system tends towards a limit cycle independent of the explicit expressions of f_1 and f_2.

3.14 The second law of thermodynamics states that the entropy of an isolated system can only increase with time. How does this relate to the notion of a Lyapunov function?

Chapter 4
Advanced Stability Theory

In the previous chapter, we studied Lyapunov analysis of autonomous systems. In many practical problems, however, we encounter non-autonomous systems. For instance, a rocket taking off is a non-autonomous system, because the parameters involved in its dynamic equations, such as air temperature and pressure, vary with time. Furthermore, as discussed earlier, determining the stability of a nominal motion for an autonomous system requires the stability analysis of an equivalent non-autonomous system around an equilibrium point. Therefore, stability analysis techniques for non-autonomous systems must be developed. This constitutes the major topic of this chapter.

After extending the concepts of stability and the Lyapunov stability theorems to non-autonomous systems, in sections 4.1 and 4.2, we discuss a number of interesting related topics in advanced stability theory. Some Lyapunov theorems for concluding *instability* of nonlinear systems are provided in section 4.3. Section 4.4 discusses a number of so-called converse theorems asserting the existence of Lyapunov functions. Besides their theoretical interest, these existence theorems can be valuable in some nonlinear system analysis and design problems. In section 4.5, we describe a very simple mathematical result, known as Barbalat's lemma, which can be conveniently used to solve some asymptotic stability problems beyond the treatment of Lyapunov stability theorems, and which we shall use extensively in chapters 8 and 9. In section 4.6, we discuss the so-called *positive real* linear systems and their unique properties, which shall be exploited later in the book, particularly in chapter 8. Section 4.7 describes passivity theory, a convenient way of interpreting, representing, and combining Lyapunov or Lyapunov-like functions. Section 4.8 discusses a special class of nonlinear systems which can be systematically treated by Lyapunov analysis. Section 4.9 studies some non-Lyapunov techniques which can be used to establish boundedness of signals in nonlinear systems. Finally, section 4.10 discusses mathematical conditions for the existence and unicity of solutions of nonlinear differential equations.

4.1 Concepts of Stability for Non-Autonomous Systems

The concepts of stability for non-autonomous systems are quite similar to those of autonomous systems. However, due to the dependence of non-autonomous system behavior on initial time t_o, the definitions of these stability concepts include t_o explicitly. Furthermore, a new concept, *uniformity*, is necessary to characterize non-autonomous systems whose behavior has a certain consistency for different values of initial time t_o. In this section, we concisely extend the stability concepts for autonomous systems to non-autonomous systems, and introduce the new concept of uniformity.

EQUILIBRIUM POINTS AND INVARIANT SETS

For non-autonomous systems, of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{4.1}$$

equilibrium points $\mathbf{x}^*$ are defined by

$$\mathbf{f}(\mathbf{x}^*, t) \equiv \mathbf{0} \qquad \forall\, t \geq t_o \tag{4.2}$$

Note that this equation must be satisfied $\forall\, t \geq t_o$, implying that the system should be able to stay at the point $\mathbf{x}^*$ all the time. For instance, one easily sees that the linear time-varying system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x} \tag{4.3}$$

has a unique equilibrium point at the origin $\mathbf{0}$ unless $\mathbf{A}(t)$ is always singular.

Example 4.1: The system

$$\dot{x} = -\frac{a(t)\,x}{1 + x^2} \tag{4.4}$$

has an equilibrium point at $x = 0$. However, the system

$$\dot{x} = -\frac{a(t)x}{1+x^2} + b(t) \tag{4.5}$$

with $b(t) \neq 0$, does not have an equilibrium point. It can be regarded as a system under external input or disturbance $b(t)$. Since Lyapunov theory is mainly developed for the stability of nonlinear systems with respect to initial conditions, such problems of forced motion analysis are more appropriately treated by other methods, such as those in section 4.9. □

The definition of invariant set is the same for non-autonomous systems as for autonomous systems. Note that, unlike in autonomous systems, a system trajectory is generally not an invariant set for a non-autonomous system (Exercise 4.1).

EXTENSIONS OF THE PREVIOUS STABILITY CONCEPTS

Let us now extend the previously defined concepts of stability, instability, asymptotic stability, and exponential stability to non-autonomous systems. The key in doing so is to properly include the initial time t_o in the definitions.

Definition 4.1 *The equilibrium point* **0** *is* *stable* *at* t_o *if for any* $R > 0$, *there exists a positive scalar* $r(R, t_o)$ *such that*

$$\|\mathbf{x}(t_o)\| < r \quad \Rightarrow \quad \|\mathbf{x}(t)\| < R \qquad \forall\, t \geq t_o \tag{4.6}$$

Otherwise, the equilibrium point **0** *is* *unstable*.

Again, the definition means that we can keep the state in a ball of arbitrarily small radius R by starting the state trajectory in a ball of sufficiently small radius r. Definition 4.1 differs from definition 3.3 in that the radius r of the initial ball may depend on the initial time t_o.

The concept of asymptotic stability can also be defined for non-autonomous systems.

Definition 4.2 *The equilibrium point* **0** *is* *asymptotically stable* *at time* t_o *if*

- *it is stable*
- $\exists\, r(t_o) > 0$ *such that* $\|\mathbf{x}(t_o)\| < r(t_o) \quad \Rightarrow \quad \|\mathbf{x}(t)\| \to \mathbf{0}$ as $t \to \infty$

Here, the asymptotic stability requires that there exists an attractive region for *every* initial time t_o. The size of attractive region and the speed of trajectory convergence may depend on the initial time t_o.

The definitions of exponential stability and global asymptotic stability are also straightforward.

Definition 4.3 *The equilibrium point* $\mathbf{0}$ *is* *exponentially stable* *if there exist two positive numbers,* α *and* λ*, such that for sufficiently small* $\mathbf{x}(t_o)$,

$$\|\mathbf{x}(t)\| \leq \alpha \|\mathbf{x}_o\|\, e^{-\lambda(t - t_o)} \qquad \forall\, t \geq t_o$$

Definition 4.4 *The equilibrium point* $\mathbf{0}$ *is* *globally asymptotically stable* *if* $\forall\ \mathbf{x}(t_o)$

$$\mathbf{x}(t) \rightarrow \mathbf{0} \qquad \text{as } t \rightarrow \infty$$

Example 4.2: A first-order linear time-varying system

Consider the first-order system

$$\dot{x}(t) = -\, a(t)\, x(t)$$

Its solution is

$$x(t) = x(t_o) \exp\left[-\int_{t_o}^{t} a(r)\, dr\right]$$

Thus, the system is stable if $a(t) \geq 0,\ \forall\, t \geq t_o$. It is asymptotically stable if $\int_o^{\infty} a(r)\, dr = +\infty$. It is exponentially stable if there exists a strictly positive number T such that $\forall\, t \geq 0$, $\int_t^{t+T} a(r)\, dr \geq \gamma$, with γ being a positive constant.

For instance,

- The system $\dot{x} = -x/(1+t)^2$ is stable (but not asymptotically stable).
- The system $\dot{x} = -\, x/(1+t)$ is asymptotically stable.
- The system $\dot{x} = -\, t x$ is exponentially stable.

Another interesting example is the system

$$\dot{x}(t) = -\frac{x}{1 + \sin x^2}$$

whose solution can be expressed as

$$x(t) = x(t_o) \exp\left[\int_{t_o}^{t} \frac{-1}{1 + \sin x^2(r)}\, dr\right]$$

Since

$$\int_{t_o}^{t} \frac{1}{1 + \sin x^2(r)}\, dr \;\geq\; \frac{t - t_o}{2}$$

the system is exponentially convergent with rate 1/2 . □

UNIFORMITY IN STABILITY CONCEPTS

The previous concepts of Lyapunov stability and asymptotic stability for non-autonomous systems both indicate the important effect of initial time. In practice, it is usually desirable for the system to have a certain uniformity in its behavior regardless of when the operation starts. This motivates us to consider the definitions of uniform stability and uniform asymptotic stability. *As we shall see in later chapters, non-autonomous systems with uniform properties have some desirable ability to withstand disturbances.* It is also useful to point out that, because the behavior of autonomous systems is independent of the initial time, all the stability properties of an autonomous system are uniform.

Definition 4.5 *The equilibrium point* **0** *is locally uniformly stable if the scalar r in Definition 4.1 can be chosen independently of t_o, i.e., if $r = r(R)$.*

The intuitive reason for introducing the concept of uniform stability is to rule out systems which are "less and less stable" for larger values of t_o. Similarly, the definition of uniform asymptotic stability also intends to restrict the effect of the initial time t_o on the state convergence pattern.

Definition 4.6 *The equilibrium point at the origin is locally uniformly asymptotically stable if*

- *it is uniformly stable*
- *There exists a ball of attraction $\mathbf{B}_{R_o}$, whose radius is independent of t_o, such that any system trajectory with initial states in $\mathbf{B}_{R_o}$ converges to* **0** *uniformly in t_o*

By uniform convergence in terms of t_o, we mean that for all R_1 and R_2 satisfying $0 < R_2 < R_1 \le R_o$, $\exists\, T(R_1, R_2) > 0$, such that, $\forall\, t_o \ge 0$,

$$\|\mathbf{x}(t_o)\| < R_1 \quad \Rightarrow \quad \|\mathbf{x}(t)\| < R_2 \qquad \forall t \ge t_o + T(R_1, R_2)$$

i.e., the state trajectory, starting from within a ball $\mathbf{B}_{R_1}$, will converge into a smaller ball $\mathbf{B}_{R_2}$ after a time period T which is independent of t_o.

By definition, uniform asymptotic stability always implies asymptotic stability. The converse is generally not true, as illustrated by the following example.

Example 4.3: Consider the first-order system

$$\dot{x} = -\frac{x}{1+t}$$

This system has the general solution

$$x(t) = \frac{1+t_o}{1+t}\, x(t_o)$$

This solution asymptotically converges to zero. But the convergence is not uniform. Intuitively, this is because a larger t_o requires a longer time to get close to the origin. □

Using Definition 4.3, one can easily prove that exponential stability always implies uniform asymptotic stability.

The concept of globally uniformly asymptotic stability can be defined by replacing the ball of attraction $\mathbf{B}_{R_o}$ by the whole state space.

4.2 Lyapunov Analysis of Non-Autonomous Systems

We now extend the Lyapunov analysis results of chapter 3 to the stability analysis of non-autonomous systems. Although many of the ideas in chapter 3 can be similarly applied to the non-autonomous case, the conditions required in the treatment of non-autonomous systems are more complicated and more restrictive. We start with the description of the direct method. We then apply the direct method to the stability analysis of linear time-varying systems. Finally, we discuss the linearization method for non-autonomous nonlinear systems.

4.2.1 Lyapunov's Direct Method for Non-Autonomous Systems

The basic idea of the direct method, *i.e.*, concluding the stability of nonlinear systems using scalar Lyapunov functions, can be similarly applied to non-autonomous systems. Besides more mathematical complexity, a major difference in non-autonomous systems is that the powerful La Salle's theorems do not apply. This drawback will partially be compensated by a simple result in section 4.5 called Barbalat's lemma.

TIME-VARYING POSITIVE DEFINITE FUNCTIONS AND DECRESCENT FUNCTIONS

When studying non-autonomous systems using Lyapunov's direct method, scalar functions with explicit time-dependence $V(t, \mathbf{x})$ may have to be used, while in autonomous system analysis time-invariant functions $V(\mathbf{x})$ suffice. We now introduce a simple definition of positive definiteness for such scalar functions.

Definition 4.7 *A scalar time-varying function $V(\mathbf{x}, t)$ is locally positive definite if $V(\mathbf{0}, t) = 0$ and there exists a time-invariant positive definite function $V_o(\mathbf{x})$ such that*

$$\forall\, t \geq t_o\,, \quad V(\mathbf{x}, t) \geq V_o(\mathbf{x}) \tag{4.7}$$

Thus, a time-variant function is locally positive definite if it *dominates a time-invariant* locally positive definite function. Globally positive definite functions can be defined similarly.

Other related concepts can be defined in the same way, in a local or a global sense. A function $V(\mathbf{x}, t)$ is negative definite if $-V(\mathbf{x}, t)$ is positive definite; $V(\mathbf{x}, t)$ is positive semi-definite if $V(\mathbf{x}, t)$ dominates a time-invariant positive semi-definite function; $V(\mathbf{x}, t)$ is negative semi-definite if $-V(\mathbf{x}, t)$ is positive semi-definite.

In Lyapunov analysis of non-autonomous systems, the concept of decrescent functions is also necessary.

Definition 4.8 *A scalar function $V(\mathbf{x}, t)$ is said to be decrescent if $V(\mathbf{0}, t) = 0$, and if there exists a time-invariant positive definite function $V_1(\mathbf{x})$ such that*

$$\forall\, t \geq 0\,, \quad V(\mathbf{x}, t) \leq V_1(\mathbf{x})$$

In other words, a scalar function $V(\mathbf{x}, t)$ is decrescent if it is *dominated by a time-invariant positive definite function.*

Example 4.4: A simple example of a time-varying positive definite function is

$$V(\mathbf{x}, t) \;=\; (1 + \sin^2 t)\,(x_1^{\,2} + x_2^{\,2})$$

because it dominates the function $V_o(\mathbf{x}) = x_1^{\,2} + x_2^{\,2}$. This function is also decrescent because it is dominated by the function $V_1(\mathbf{x}) = 2(x_1^{\,2} + x_2^{\,2})$. □

Given a time-varying scalar function $V(\mathbf{x}, t)$, its derivative along a system trajectory is

$$\frac{dV}{dt} \;=\; \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}}\,\dot{\mathbf{x}} \;=\; \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}} f(\mathbf{x}, t) \tag{4.8}$$

LYAPUNOV THEOREM FOR NON-AUTONOMOUS SYSTEM STABILITY

The main Lyapunov stability results for non-autonomous systems can be summarized by the following theorem.

Theorem 4.1 (Lyapunov theorem for non-autonomous systems)

Stability: *If, in a ball* $\mathbf{B}_{R_o}$ *around the equilibrium point* $\mathbf{0}$*, there exists a scalar function* $V(\mathbf{x}, t)$ *with continuous partial derivatives such that*

1. V is positive definite

2. $\dot{V}$ *is negative semi-definite*

then the equilibrium point $\mathbf{0}$ *is stable in the sense of Lyapunov.*

Uniform stability and uniform asymptotic stability: *If, furthermore,*

3. V is decrescent

then the origin is uniformly stable. If condition 2 is strengthened by requiring that $\dot{V}$ *be negative definite, then the equilibrium point is uniformly asymptotically stable.*

Global uniform asymptotic stability: *If the ball* B_{R_o} *is replaced by the whole state space, and condition 1, the strengthened condition 2, condition 3, and the condition*

4. $V(\mathbf{x},t)$ *is radially unbounded*

are all satisfied, then the equilibrium point at $\mathbf{0}$ *is globally uniformly asymptotically stable.*

Similarly to the case of autonomous systems, if, in a certain neighborhood of the equilibrium point, V is positive definite and $\dot{V}$, its derivative along the system trajectories, is negative semi-definite, then V is called a Lyapunov function for the non-autonomous system.

The proof of this important theorem, which we now detail, is rather technical. Hurried readers may skip it in a first reading, and go directly to Example 4.5.

In order to prove the above theorem, we first translate the definitions of positive definite functions and decrescent functions in terms of the so-called class-K functions.

Definition 4.9 *A continuous function* $\alpha: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ *is said to be of* <u>*class K*</u> *(or to belong to class K), if*

- $\alpha(0) = 0$

- $\alpha(p) > 0 \quad \forall p > 0$

- α *is non-decreasing*

The following lemma indicates the relation of positive definite and decrescent functions to class K functions.

Lemma 4.1: *A function $V(\mathbf{x}, t)$ is locally (or globally) positive definite if, and only if, there exists a function α of class K such that $V(\mathbf{0}, t) = 0$ and*

$$V(\mathbf{x}, t) \geq \alpha(\|\mathbf{x}\|) \tag{4.9}$$

$\forall\, t \geq 0$ *and* $\forall\, \mathbf{x} \in \mathbf{B}_{R_o}$ *(or the whole state space).*

A function $V(\mathbf{x}, t)$ is locally (or globally) decrescent if and only if there exists a class K function β such that $V(\mathbf{0}, t) = 0$ and

$$V(\mathbf{x}, t) \leq \beta(\|\mathbf{x}\|) \tag{4.10}$$

$\forall\, t \geq 0$ *and* $\forall\, \mathbf{x} \in \mathbf{B}_{R_0}$ *(or in the whole state space).*

Proof: Let us prove the positive definite function part first. Sufficiency is obvious from the definition, because $\alpha(\|\mathbf{x}\|)$ itself is a scalar time-invariant positive definite function. We now consider necessity, *i.e.*, assume that there exists a time-invariant positive function $V_o(\mathbf{x})$ such that $V(\mathbf{x}, t) \geq V_o(\mathbf{x})$, and show that a function α of class K exists such that (4.9) holds. Let us define

$$\alpha(p) = \inf_{p \,\leq\, \|\mathbf{x}\| \,\leq\, R} V_o(\mathbf{x}) \tag{4.11}$$

Then, $\alpha(0) = 0$, α is continuous and non-decreasing. Because $V_o(\mathbf{x})$ is a continuous function and non-zero except at $\mathbf{0}$, $\alpha(p) > 0$ for $p > 0$. Therefore, α is a class K function. Because of (4.11), (4.9) is satisfied.

The second part of the lemma can be proven similarly, with the function β defined by

$$\beta(p) = \sup_{0 \leq\, \|\mathbf{x}\| \,\leq\, p} V_1(\mathbf{x}) \tag{4.12}$$

where $V_1(\mathbf{x})$ is the time-invariant positive function in Definition 4.8. □

Given the above lemma, we can now restate Theorem 4.1 as follows:

Theorem 4.1 *Assume that, in a neighborhood of the equilibrium point $\mathbf{0}$, there exists a scalar function $V(\mathbf{x}, t)$ with continuous first order derivatives and a class-K function α such that, $\forall \mathbf{x} \neq \mathbf{0}$*

1. $V(\mathbf{x}, t) \geq \alpha(\|\mathbf{x}\|) > 0$

2a. $\dot{V}(\mathbf{x}, t) \leq 0$

then the origin $\mathbf{0}$ is Lyapunov stable. If, furthermore, there is a scalar class-K function β such that

3. $V(\mathbf{x}, t) \leq \beta(\|\mathbf{x}\|)$

then **0** *is uniformly stable. If conditions 1 and 3 are satisfied and condition 2a is replaced by condition 2b*

$$2b.\ \dot{V} \leq -\gamma(\|\mathbf{x}\|) < 0$$

with γ *being another class-K function, then* **0** *is uniformly asymptotically stable. If conditions 1, 2b and 3 are satisfied in the whole state space, and*

$$\lim_{\mathbf{x} \to \infty} \alpha(\|\mathbf{x}\|) \to \infty$$

then **0** *is globally uniformly asymptotically stable.*

Proof: We derive the three parts of the theorem in sequence.

Lyapunov stability: To establish Lyapunov stability, we must show that given $R > 0$, there exists $r > 0$ such that (4.6) is satisfied. Because of conditions 1 and 2a,

$$\alpha(\|\mathbf{x}(t)\|) \leq V[\mathbf{x}(t), t] \leq V[\mathbf{x}(t_o), t_o] \qquad \forall t \geq t_o \tag{4.13}$$

Because V is continuous in terms of $\mathbf{x}$ and $V(\mathbf{0}, t_o) = 0$, we can find r such that

$$\|\mathbf{x}(t_o)\| < r \quad \Rightarrow \quad V(\mathbf{x}(t_o), t_o) < \alpha(R)$$

This means that if $\|\mathbf{x}(t_o)\| < r$, then $\alpha(\|\mathbf{x}(t)\|) < \alpha(R)$, and, accordingly, $\|\mathbf{x}(t)\| < R$, $\forall t \geq t_o$.

Uniform stability and uniform asymptotic stability: From conditions 1 and 3,

$$\alpha(\|\mathbf{x}(t)\|) \leq V(\mathbf{x}(t), t) \leq \beta(\|\mathbf{x}(t)\|)$$

For any $R > 0$, there exists $r(R) > 0$ such that $\beta(r) < \alpha(R)$ (Figure 4.1). Let the initial condition $\mathbf{x}(t_o)$ be chosen such that $\|\mathbf{x}(t_o)\| < r$. Then

$$\alpha(R) > \beta(r) \geq V[\mathbf{x}(t_o), t_o] \geq V[\mathbf{x}(t), t] \geq \alpha(\|\mathbf{x}(t)\|)$$

This implies that

$$\forall\ t \geq t_o\ , \quad \|\mathbf{x}(t)\| < R$$

Uniform stability is asserted because r is independent of t_o .

In establishing uniform asymptotic stability, the basic idea is that if $\mathbf{x}$ does not converge to the origin, then it can be shown that there is a positive number a such that $-\dot{V}\,[\mathbf{x}(t), t] \geq a > 0$. This implies that

$$V[\mathbf{x}(t), t] - V[\mathbf{x}(t_o), t_o] = \int_{t_o}^{t} \dot{V}\, dt \leq -(t - t_0)\, a$$

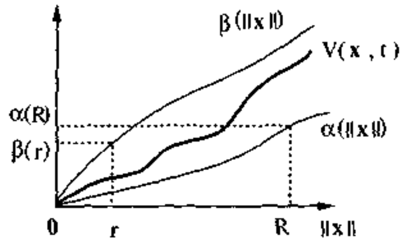


Figure 4.1 : A positive definite and decrescent function

and thus, that

$$0 \leq V[\mathbf{x}(t), t] \leq V[\mathbf{x}(t_o), t_o] - (t - t_0)\, a$$

which leads to a contradiction for large t. Let us now detail the proof.

Let $\|\mathbf{x}(t_o)\| \leq r$, with r obtained as before. Let μ be be any positive constant such that $0 < \mu < \|\mathbf{x}(t_o)\|$. We can find another positive constant $\delta(\mu)$ such that $\beta(\delta) < \alpha(\mu)$. Define $\varepsilon = \gamma(\delta)$ and set

$$T = T(\mu, r) = \frac{\beta}{\varepsilon}$$

Then, if $\|\mathbf{x}(t)\| > \mu$ for all t in the period $t_o \leq t \leq t_1 \equiv t_o + T$, we have

$$0 < \alpha(\mu) \leq V[\mathbf{x}(t_1), t_1] \leq V[\mathbf{x}(t_o, t_o] - \int_{t_o}^{t_1} \gamma(\|\mathbf{x}(s)\|)\, ds \leq V[\mathbf{x}(t_o), t_o)] - \int_{t_o}^{t_1} \gamma(\delta)\, ds$$

$$\leq V[\mathbf{x}(t_o), t_o] - (t_1 - t_o)\varepsilon \leq \beta(r) - T\varepsilon = 0$$

This is a contradiction, and so there must exist $t_2 \in [t_o, t_1]$ such that $\|\mathbf{x}(t_2)\| \leq \delta$. Thus, for all $t \geq t_2$,

$$\alpha(\|\mathbf{x}(t)\|) \leq V[\mathbf{x}(t), t] \leq V[\mathbf{x}(t_2), t_2] \leq \beta(\delta) < \alpha(\mu)$$

As a result,

$$\|\mathbf{x}(t)\| < \mu \qquad \forall t \geq t_o + T \geq t_2$$

which shows uniform asymptotic stability.

Global uniform asymptotic stability: Since $\alpha(\cdot)$ is radially unbounded, R can be found such that $\beta(r) < \alpha(R)$ for any r. In addition, r can be made arbitrarily large. Hence, the origin $\mathbf{x} = \mathbf{0}$ is globally uniformly asymptotically stable. □

Example 4.5: Global Asymptotic Stability

Consider the system defined by

$$\dot{x}_1(t) = -x_1(t) - e^{-2t}\, x_2(t)$$

$$\dot{x}_2(t) = x_1(t) - x_2(t)$$

To determine the stability of the equilibrium point at **0**, let us choose the following scalar function

$$V(\mathbf{x}, t) = x_1^{\,2} + (1 + e^{-2t})\, x_2^{\,2}$$

This function is positive definite, because it dominates the time-invariant positive function $x_1^{\,2} + x_2^{\,2}$. It is also decrescent, because it is dominated by the time-invariant positive definite function $x_1^{\,2} + 2x_2^{\,2}$. Furthermore,

$$\dot{V}(\mathbf{x}, t) = -2[\, x_1^{\,2} - x_1 x_2 + x_2^{\,2}(1 + 2e^{-2t})\,]$$

This shows that

$$\dot{V} \leq -2(x_1^{\,2} - x_1 x_2 + x_2^{\,2}) = -(x_1 - x_2)^2 - x_1^{\,2} - x_2^{\,2}$$

Thus, $\dot{V}$ is negative definite, and therefore, the point **0** is globally asymptotically stable. □

Stability results for non-autonomous systems can be less intuitive than those for autonomous systems, and therefore, particular care is required in applying the above theorem, as we now illustrate with two simple examples.

For autonomous systems, the origin is guaranteeed to be asymptotically stable if V is positive definite and $\dot{V}$ is negative definite. Therefore, one might be tempted to conjecture that the same conditions are sufficient to guarantee the asymptotic stability of the system. However, this intuitively appealing statement is *incorrect*, as demonstrated by the following counter-example.

Example 4.6: Importance of the decrescence condition

Let $g(t)$ be a continuously-differentiable function which coincides with the function $e^{-t/2}$ except around some peaks where it reaches the value 1. Specifically, $g^2(t)$ is shown in Figure 4.2. There is a peak for each integer value of t. The width of the peak corresponding to abcissa n is assumed to be smaller than $(1/2)^n$. The infinite integral of g^2 thus satisfies

$$\int_o^{\infty} g^2(r)\,dr \;<\; \int_o^{\infty} e^{-r}\,dr \;+\; \sum_{n=1}^{\infty} \frac{1}{2^n} \;=\; 2$$

and therefore, the scalar function

$$V(\mathbf{x}, t) = \frac{x^2}{g^2(t)} [\, 3 - \int_o^t g^2(r)\, dr \,] \tag{4.14}$$

is positive definite ($V(\mathbf{x}, t) > x^2$).

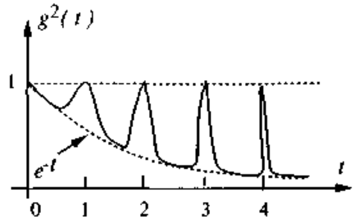


Figure 4.2 : The function $g^2(t)$

Now consider the first-order differential equation

$$\dot{x} = \frac{\dot{g}(t)}{g(t)}\, x \tag{4.15}$$

If we choose $V(\mathbf{x}, t)$ in (4.14) as the Lyapunov function candidate, we easily find that

$$\dot{V} = -\, x^2$$

i.e., $\dot{V}$ *is negative definite.* Yet, the general solution of (4.15) is

$$x(t) = \frac{g(t)}{g(t_o)}\, x(t_o)$$

and hence *the origin is not asymptotically stable.* □

Since the positive definiteness of V and the negative semi-definiteness of $\dot{V}$ are already sufficient to guarantee the Lyapunov stability of the origin, one may wonder what additional property the negative definiteness of $\dot{V}$ does provide. It can be shown that, if $\dot{V}$ is negative definite, then an infinite sequence t_i's ($i = 1, 2, ...$) can be found such that the corresponding state values $\mathbf{x}(t_i)$ converge to zero as $i \to \infty$ (a result of mostly theoretical interest).

Another illustration that care is required before jumping to conclusions involves the following interesting second-order dynamics

$$\ddot{x} + c(t)\,\dot{x} + k_o x = 0 \tag{4.16}$$

which can represent a mass-spring-damper system (of mass 1), where $c(t) \geq 0$ is a time-varying damping coefficient, and k_o is a spring constant. Physical intuition may suggest that the equilibrium point (0,0) is asymptotically stable as long as the damping

$c(t)$ remains larger than a strictly positive constant (implying constant dissipation of energy), as is the case for autonomous nonlinear mass-spring-damper systems. However, *this is not necessarily true.* Indeed, consider the system

$$\ddot{x} + (2+e^t)\dot{x} + x = 0$$

One easily verifies that, for instance, with the initial condition $x(0) = 2$, $\dot{x}(0) = -1$, the solution is $x(t) = 1 + e^{-t}$, which tends to $x = 1$ instead! Here the damping increases so fast that the system gets "stuck" at $x = 1$.

Let us study the asymptotic stability of systems of the form of (4.16), using a Lyapunov analysis.

Example 4.7: Asymptotic stability with time-varying damping

Lyapunov stability of the system (although not its asymptotic stability) can be easily established using the mechanical energy of the system as a Lyapunov function. Let us now use a different Lyapunov function to determine sufficient conditions for the asymptotic stability of the origin for the system (4.16). Consider the following positive definite function

$$V(\mathbf{x}, t) = \frac{(\dot{x} + \alpha x)^2}{2} + \frac{b(t)}{2} x^2$$

where α is any positive constant smaller than $\sqrt{k_o}$, and

$$b(t) = k_o - \alpha^2 + \alpha c(t)$$

$\dot{V}$ can be easily computed as

$$\dot{V} = [\alpha - c(t)]\,\dot{x}^2 + \frac{\alpha}{2}\,[\dot{c}(t) - 2\,k_o]\,x^2$$

Thus, if there exist positive numbers α and β such that

$$c(t) > \alpha > 0 \qquad \dot{c}(t) \leq \beta < 2\,k_o$$

then $\dot{V}$ is negative definite. Assuming in addition that $c(t)$ is upper bounded (guaranteeing the decrescence of V), the above conditions imply the asymptotic convergence of the system.

It can be shown [Rouche, *et al.*, 1977] that, actually, the technical assumption that $c(t)$ is upper bounded is not necessary. Thus, for instance, the system

$$\ddot{x} + (2+8\,t)\dot{x} + 5\,x = 0$$

is asymptotically stable. □

4.2.2 Lyapunov Analysis of Linear Time-Varying Systems

None of the standard approaches for analyzing linear time-invariant systems (*e.g.*, eigenvalue determination) applies to linear time-varying systems. Thus, it is of interest to consider the application of Lyapunov's direct method for studying the stability of linear time-varying systems. Indeed, a number of results are available in this regard. In addition, in view of the relation between the stability of a non-autonomous system and that of its (generally time-varying) linearization, to be discussed in section 4.2.3, such results on linear time-varying systems can also be of practical importance for local stability analysis of nonlinear non-autonomous systems.

Consider linear time-varying systems of the form

$$\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x} \tag{4.17}$$

Since LTI systems are asymptotically stable if their eigenvalues all have negative real parts, one might be tempted to conjecture that system (4.17) will be stable if at *any* time $t \geq 0$, the eigenvalues of $A(t)$ all have negative real parts. If this were indeed the case, it would make the analysis of linear time-varying systems very easy. However, this conjecture is *not true*.

Consider for instance the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4.18}$$

The eigenvalues of the matrix $\mathbf{A}(t)$ are both equal to -1 at all times. Yet, solving first for x_2 and then substituting in the $\dot{x}_1$ equation, one sees that the system verifies

$$x_2 = x_2(0)\,e^{-t} \qquad\qquad \dot{x}_1 + x_1 = x_2(0)\,e^{t}$$

and therefore is unstable, since x_1 can be viewed as the output of a first-order filter whose input $x_2(0)\,e^t$ tends to infinity.

A simple result, however, is that the time-varying system (4.17) *is* asymptotically stable if the eigenvalues of the *symmetric* matrix $\mathbf{A}(t) + \mathbf{A}^T(t)$ (all of which are real) remain strictly in the left-half complex plane

$$\exists\, \lambda > 0\,,\ \forall\, i\,,\ \forall\, t \geq 0\,,\ \lambda_i(\mathbf{A}(t) + \mathbf{A}^T(t)) \leq -\lambda \tag{4.19}$$

This can be readily shown using the Lyapunov function $V = \mathbf{x}^T\mathbf{x}$, since

$$\dot{V} = \mathbf{x}^T\dot{\mathbf{x}} + \dot{\mathbf{x}}^T\mathbf{x} = \mathbf{x}^T\,(\mathbf{A}(t) + \mathbf{A}^T(t))\,\mathbf{x} \leq -\,\lambda\,\mathbf{x}^T\mathbf{x} = -\,\lambda\,V$$

so that

$$\forall\, t \geq 0\,, \quad 0 \leq \mathbf{x}^T\mathbf{x} = V(t) \leq V(0)\,e^{-\lambda t}$$

and therefore $\mathbf{x}$ tends to zero exponentially.

Of course, the above result also applies in case the matrix A depends explicitly on the state. It is also important to notice that the result provides a *sufficient* condition for asymptotic stability (some asymptotically stable systems do not verify (4.19), see Exercise 4.8).

A large number of more specialized theorems are available to determine the stability of classes of linear time-varying systems. We now give two examples of such results. The first result concerns "perturbed" linear systems, *i.e.*, systems of the form (4.17), where the matrix $\mathbf{A}(t)$ is a sum of a constant stable matrix and some "small" time-varying matrix. The second result is a more technical property of systems such that the matrix $\mathbf{A}(t)$ maintains all its eigenvalues in the left-half plane *and* satisfies certain smoothness conditions .

Perturbed linear systems

Consider a linear time-varying system of the form

$$\dot{\mathbf{x}} = \Big(\mathbf{A}_1 + \mathbf{A}_2(t)\Big)\,\mathbf{x} \tag{4.20}$$

where the matrix $\mathbf{A}_1$ is constant and Hurwitz (*i.e.*, has all its eigenvalues strictly in the left-half plane), and the time-varying matrix $\mathbf{A}_2(t)$ is such that

$$\mathbf{A}_2(t) \;\rightarrow\; \mathbf{0} \;\text{ as }\; t \rightarrow \infty$$

and

$$\int_0^\infty \|\,\mathbf{A}_2(t)\,\|\,dt \;<\; \infty \qquad (i.e, \text{the integral exists and is finite})$$

Then the system (4.20) is globally exponentially stable.

Example 4.8: Consider the system

$$\dot{x}_1 = -\Big(5 + x_2^{\,5} + x_3^{\,8}\Big)\,x_1$$

$$\dot{x}_2 = -\,x_2 + 4\,x_3^{\,2}$$

$$\dot{x}_3 = -\,(2 + \sin t)\,x_3$$

Since x_3 tends to zero exponentially, so does x_3^2, and therefore, so does x_2. Applying the above result to the first equation, we conclude that the system is globally exponentially stable. □

Sufficient smoothness conditions on the A(t) matrix

Consider the linear system (4.17), and assume that at *any* time $t \geq 0$, the eigenvalues of $A(t)$ all have negative real parts

$$\exists\, \alpha > 0\,,\ \forall\, i\,,\ \forall\, t \geq 0\,,\ \lambda_i[\mathbf{A}(t)] \ \leq\ -\alpha \tag{4.21}$$

If, *in addition*, the matrix $\mathbf{A}(t)$ remains *bounded*, and

$$\int_o^{\infty} \mathbf{A}^T(t)\,\mathbf{A}(t)\,dt\ <\ \infty \qquad (i.e, \text{ the integral exists and is finite})$$

then the system is globally exponentially stable.

4.2.3 * The Linearization Method for Non-Autonomous Systems

Lyapunov's linearization method can also be developed for non-autonomous systems. Let a non-autonomous system be described by (4.1) and $\mathbf{0}$ be an equilibrium point. Assume that $\mathbf{f}$ is continuously differentiable with respect to $\mathbf{x}$. Let us denote

$$\mathbf{A}(t)\ =\ \left(\frac{\partial\,\mathbf{f}}{\partial\,\mathbf{x}}\right)_{\mathbf{x}=\mathbf{0}} \tag{4.22}$$

Then for any *fixed* time t (*i.e.*, regarding t as a parameter), a Taylor expansion of $\mathbf{f}$ leads to

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}_{h.o.t.}(\mathbf{x},t)$$

If $\mathbf{f}$ can be well approximated by $\mathbf{A}(t)\mathbf{x}$ for any time t, *i.e.*,

$$\lim_{\|\mathbf{x}\| \to 0}\ \sup \frac{\|\ \mathbf{f}_{h.o.t}(\mathbf{x},t)\ \|}{\|\mathbf{x}\|} = 0 \qquad \forall\, t \geq 0 \tag{4.23}$$

then the system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x} \tag{4.24}$$

is said to be the linearization (or linear approximation) of the nonlinear non-autonomous system (4.1) around equilibrium point $\mathbf{0}$.

Note that

• The Jacobian matrix **A** thus obtained from a non-autonomous nonlinear system is generally time-varying, contrary to what happens for autonomous nonlinear systems. But in some cases **A** is constant. For example, the nonlinear system $\dot{x} = -x + x^2/t$ leads to the linearized system $\dot{x} = -x$.

• Our later results require that the uniform convergence condition (4.23) be satisfied. Some non-autonomous systems may not satisfy this condition, and Lyapunov's linearization method cannot be used for such systems. For example, (4.23) is *not* satisfied for the system $\dot{x} = -x + tx^2$.

Given a non-autonomous system satisfying condition (4.23), we can assert its (local) stability if its linear approximation is uniformly asymptotically stable, as stated in the following theorem:

Theorem 4.2 *If the linearized system (with condition (4.23) satisfied) is uniformly asymptotically stable, then the equilibrium point* **0** *of the original non-autonomous system is also uniformly asymptotically stable.*

Note that the linearized time-varying system must be *uniformly* asymptotically stable in order to use this theorem. If the linearized system is only asymptotically stable, no conclusion can be drawn about the stability of the original nonlinear system. Counter-examples can easily verify this point.

Unlike Lyapunov's linearization method for autonomous systems, the above theorem does not relate the instability of the linearized time-varying system to that of the nonlinear system. There does exist a simple result which infers the instability of a non-autonomous system from that of its linear approximation, but it is applicable only to non-autonomous systems whose linear approximations are time-invariant.

Theorem 4.3 *If the Jacobian matrix* $\mathbf{A}(t)$ *is constant,* $\mathbf{A}(t) = \mathbf{A}_o$*, and if (4.23) is satisfied, then the instability of the linearized system implies that of the original non-autonomous nonlinear system, i.e., (4.1) is unstable if one or more of the eigenvalues of* $\mathbf{A}_o$ *has a positive real part.*

4.3 * Instability Theorems

The preceding study of Lyapunov's direct method is concerned with providing sufficient conditions for stability of nonlinear systems. This section provides some instability theorems based on Lyapunov's direct method.

Note that, for *autonomous* systems, one might think that the conclusive results provided by Lyapunov's linearization method are sufficient for the study of instability.

However, in some cases, instability theorems based on the direct method may be advantageous. Indeed, if the linearization method fails (*i.e.*, if the linearized system is marginally stable), these theorems may be used to determine the instability of the nonlinear system. Another advantage may be convenience, since the theorems do not require linearization of the system equations.

We state the theorems directly in a non-autonomous setting. For autonomous systems, the conditions simplify in a straightforward fashion.

Theorem 4.4 (First instability theorem) *If, in a certain neighborhood* $\mathbf{\Omega}$ *of the origin, there exists a continuously differentiable, decrescent scalar function* $V(\mathbf{x}, t)$ *such that*

- $V(\mathbf{0}, t) = 0 \quad \forall\, t \geq t_o$
- $V(\mathbf{x}, t_o)$ *can assume strictly positive values arbitrarily close to the origin*
- $\dot{V}(\mathbf{x}, t)$ *is positive definite (locally in* $\mathbf{\Omega}$*)*

then the equilibrium point $\mathbf{0}$ *at time* t_o *is unstable.*

Note that the second condition is weaker than requiring the positive definiteness of V. For example, the function $V(\mathbf{x}) = x_1^{\,2} - x_2^{\,2}$ is obviously not positive definite, but it can assume positive values arbitrarily near the origin ($V(\mathbf{x}) = x_1^{\,2}$ along the line $x_2 = 0$).

Example 4.9: Consider the system

$$\dot{x}_1 = 2x_2 + x_1(x_1^{\,2} + 2x_2^{\,4}) \tag{4.26}$$

$$\dot{x}_2 = -2x_1 + x_2(x_1^{\,2} + x_2^{\,4}) \tag{4.27}$$

Linearization of this system yields $\dot{x}_1 = 2x_2$ and $\dot{x}_2 = -2x_1$. The eigenvalues of this system are $+2j$ and $-2j$, indicating the inability of Lyapunov's linearization method for this system. However, if we take

$$V = \frac{1}{2}(x_1^{\,2} + x_2^{\,2})$$

its derivative is

$$\dot{V} = (x_1^{\,2} + x_2^{\,2})(x_1^{\,2} + x_2^{\,4})$$

Because of the positive definiteness of V and $\dot{V}$, the above theorem indicates the instability of the system. □

Theorem 4.5 (Second instability theorem) *If, in a certain neighborhood Ω of the origin, there exists a continuously differentiable, decrescent scalar function $V(\mathbf{x}, t)$ satisfying*

- *$V(\mathbf{0}, t_o) = 0$ and $V(\mathbf{x}, t_o)$ can assume strictly positive values arbitrarily close to the origin*
- *$\dot{V}(\mathbf{x}, t) - \lambda V(\mathbf{x}, t) \geq 0 \quad \forall\, t \geq t_o \quad \forall\, \mathbf{x} \in \Omega$*

with λ being a strictly positive constant, then the equilibrium point $\mathbf{0}$ at time t_o is unstable.

Example 4.10: Consider the system described by

$$\dot{x}_1 = x_1 + 3x_2 \sin^2 x_2 + 5x_1 x_2^2 \sin^2 x_1 \tag{4.28}$$

$$\dot{x}_2 = 3x_1 \sin^2 x_2 + x_2 - 5x_1^2 x_2 \cos^2 x_1 \tag{4.29}$$

Let us consider the function $V(\mathbf{x}) = (1/2)(x_1^2 - x_2^2)$, which was shown earlier to assume positive values arbitrarily near the origin. Its derivative is

$$\dot{V} = x_1^2 - x_2^2 + 5x_1^2 x_2^2 = 2V + 5x_1^2 x_2^2$$

Thus, the second instability theorem shows that the equilibrium point at the origin is unstable. Of course, in this particular case, the instability could be predicted more easily by the linearization method. □

In order to apply the above two theorems, $\dot{V}$ is required to satisfy certain conditions at all points in the neighborhood Ω. The following theorem (Cetaev's theorem) replaces theses conditions by a boundary condition on a subregion in Ω.

Theorem 4.6 (Third instability theorem) *Let Ω be a neighborhood of the origin. If there exists a scalar function $V(\mathbf{x}, t)$ with continuous first partial derivatives, decrescent in Ω, and a region Ω_1 in Ω, such that*

- *$V(\mathbf{x}, t)$ and $\dot{V}(\mathbf{x}, t)$ are positive definite in Ω_1*
- *The origin is a boundary point of Ω_1*
- *At the boundary points of Ω_1 within Ω, $V(\mathbf{x}, t) = 0$ for all $t \geq t_o$*

then the equilibrium point $\mathbf{0}$ at time t_o is unstable.

The geometrical meaning of this theorem can be seen from Figure 4.3. Let us illustrate the use of this theorem on a simple example.

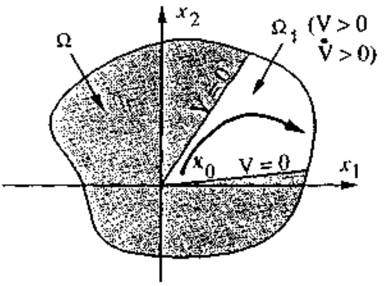


Figure 4.3 : Geometrical interpretation of the third instability theorem

Example 4.11: Consider the system

$$\dot{x}_1 = x_1^{\,2} + x_2^{\,3}$$

$$\dot{x}_2 = -x_2 + x_1^{\,3}$$

The linearization of this system leads to a pole at the origin and a pole at -1. Therefore, Lyapunov's linearization method cannot be used to determine the stability of this nonlinear system. Now let us take the function $V = x_1 - x_2^{\,2}/2$. Its derivative is

$$\dot{V} = \dot{x}_1 - x_2\dot{x}_2 = x_1^{\,2} + x_2^{\,2} + x_2^{\,3} - x_1^{\,3}x_2$$

Examining V and $\dot{V}$ and using Cetaev's theorem, one can show the instability of the origin. □

4.4 * Existence of Lyapunov Functions

In the previous Lyapunov theorems, the existence of Lyapunov functions is always assumed, and the objective is to deduce the stability properties of the systems from the properties of the Lyapunov functions. In view of the common difficulty in finding Lyapunov functions, one may naturally wonder whether Lyapunov functions always exist for stable systems. A number of interesting results concerning the existence of Lyapunov functions, called *converse Lyapunov theorems*, have been obtained in this regard. For many years, these theorems were thought to be of no practical value because, like the previously described theorems, they do not tell us how to generate Lyapunov functions for a system to be analyzed, but only represent comforting reassurances in the search for Lyapunov functions. In the past few years, however, there has been a resurgence of interest in these results. The reason is that a subsystem of a nonlinear system may be known to possess some stability properties, and the converse theorems allow us to construct a Lyapunov function for the subsystem, which may subsequently lead to the generation of a Lyapunov function for the whole system. In particular, the converse theorems can be used in connection with stability analysis of feedback linearizable systems and robustness analysis of adaptive control systems.

THE CONVERSE THEOREMS

There exists a converse theorem for essentially every Lyapunov stability theorem (stability, uniform stability, asymptotic stability, uniform asymptotic stability, global uniform asymptotic stability and instability). We now present three of the converse theorems.

Theorem 4.7 (stability) *If the origin of (4.1) is stable, there exists a positive definite function $V(\mathbf{x}, t)$ with a non-positive derivative.*

This theorem indicates the existence of a Lyapunov function for every stable system.

Theorem 4.8 (uniform asymptotic stability) *If the equilibrium point at the origin is uniformly asymptotically stable, there exists a positive definite and decrescent function $V(\mathbf{x}, t)$ with a negative definite derivative.*

This theorem is theoretically important because it will later be useful in establishing robustness of uniform asymptotic stability to persistent disturbance.

The next theorem on exponential stability has more practical value than the second theorem, because its use may allow us to explicitly estimate convergence rates in some nonlinear systems.

Theorem 4.9 (exponential stability) *If the vector function $\mathbf{f}(\mathbf{x}, t)$ in (4.1) has continuous and bounded first partial derivatives with respect to $\mathbf{x}$ and t, for all $\mathbf{x}$ in a ball $\mathbf{B}_r$ and for all $t \geq 0$, then the equilibrium point at the origin is exponentially stable if, and only if, there exists a function $V(\mathbf{x}, t)$ and strictly positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that $\forall\ \mathbf{x} \in \mathbf{B}_r$, $\forall\ t \geq 0$*

$$\alpha_1 \|\mathbf{x}\|^2 \leq V(\mathbf{x}, t) \leq \alpha_2 \|\mathbf{x}\|^2 \tag{4.30}$$

$$\dot{V} \leq -\alpha_3 \|\mathbf{x}\|^2 \tag{4.31}$$

$$\left\| \frac{\partial V}{\partial \mathbf{x}} \right\| \leq \alpha_4 \|\mathbf{x}\| \tag{4.32}$$

The proofs of the converse theorems typically assume that the solution of the system is available, and then construct a Lyapunov function based on the assumed solution [Hahn, 1967]. Proof of Theorem 4.9 can be found in [Bodson, 1986; Sastry and Bodson, 1989].

4.5 Lyapunov-Like Analysis Using Barbalat's Lemma

For autonomous systems, the invariant set theorems are powerful tools to study stability, because they allow asymptotic stability conclusions to be drawn even when $\dot{V}$ is only negative *semi*-definite. However, the invariant set theorems are not applicable to non-autonomous systems. Therefore, *asymptotic* stability analysis of non-autonomous systems is generally much harder than that of autonomous systems, since it is usually very difficult to find Lyapunov functions with a *negative definite* derivative. An important and simple result which partially remedies this situation is Barbalat's lemma. Barbalat's lemma is a purely mathematical result concerning the asymptotic properties of functions and their derivatives. When properly used for dynamic systems, particularly non-autonomous systems, it may lead to the satisfactory solution of many asymptotic stability problems.

4.5.1 Asymptotic Properties of Functions and Their Derivatives

Before discussing Barbalat's lemma itself, let us clarify a few points concerning the asymptotic properties of functions and their derivatives. Given a differentiable function f of time t, the following three facts are important to keep in mind.

- $\dot{f} \rightarrow 0 \not\Rightarrow f$ converges

  The fact that $\dot{f}(t) \rightarrow 0$ does <u>not</u> imply that $f(t)$ has a limit as $t \rightarrow \infty$.

  Geometrically, a diminishing derivative means flatter and flatter slopes. However, it does not necessarily imply that the function approaches a limit. Consider, for instance, the rather benign function $f(t) = \sin(\log t)$. While

  $$\dot{f}(t) = \frac{\cos(\log t)}{t} \rightarrow 0 \text{ as } t \rightarrow \infty$$

  the function $f(t)$ keeps oscillating (slower and slower). The function $f(t)$ may even be unbounded, as with $f(t) = \sqrt{t}\ \sin(\log t)$. Note that functions of the form $\log t$, $\sin t$, $e^{\alpha t}$, and combinations thereof, are quite easy to find in dynamic system responses.

- f converges $\not\Rightarrow \dot{f} \rightarrow 0$

  The fact that $f(t)$ has a finite limit as $t \rightarrow \infty$ does <u>not</u> imply that $\dot{f}(t) \rightarrow 0$.

  For instance, while the function $f(t) = e^{-t}\sin(e^{2t})$ tends to zero, its derivative $\dot{f}$ is unbounded. Note that this is not linked to the frequent sign changes of the function. Indeed, with $f(t) = e^{-t} \sin^2(e^{2t}) \geq 0$, $\dot{f}$ is still unbounded.

- If f is lower bounded and decreasing ($\dot{f} \leq 0$), then it converges to a limit.

  This is a standard result in calculus. However, it does not say whether the slope of the curve will diminish or not.

4.5.2 Barbalat's Lemma

Now, given that a function tends towards a finite limit, what additional requirement can guarantee that its derivative actually converges to zero? Barbalat's lemma indicates that the derivative itself should have some smoothness. More precisely, we have

Lemma 4.2 (Barbalat) *If the differentiable function $f(t)$ has a finite limit as $t \to \infty$, and if $\dot{f}$ is uniformly continuous, then $\dot{f}(t) \to 0$ as $t \to \infty$.*

Before proving this result, let us define what we mean by *uniform* continuity. Recall that a function $g(t)$ is continuous on $[0, \infty)$ if

$$\forall\, t_1 \geq 0, \forall\, R > 0\,, \exists\, \eta(R, t_1) > 0\,, \forall\, t \geq 0, \quad |t - t_1| < \eta \quad => \quad |g(t) - g(t_1)| < R$$

A function g is said to be <u>*uniformly continuous*</u> on $[0, \infty)$ if

$$\forall\, R > 0\,, \exists\, \eta(R) > 0\,, \forall\, t_1 \geq 0, \forall\, t \geq 0, \quad |t - t_1| < \eta \quad => \quad |g(t) - g(t_1)| < R$$

In other words, g is *uniformly* continuous if one can always find an η which does not depend on the specific point t_1 – and in particular, such that η does not shrink as $t_1 \to \infty$, as shall be important when proving the lemma. Note that t and t_1 play a symmetric role in the definition of uniform continuity.

Uniform continuity of a function is often awkward to assert directly from the above definition. A more convenient approach is to examine the function's derivative. Indeed, a very simple *sufficient* condition for a differentiable function to be uniformly continuous is that *its derivative be bounded*. This can be easily seen from the finite difference theorem

$$\forall\, t, \forall\, t_1\,, \quad \exists\, t_2 \text{ (between } t \text{ and } t_1) \text{ such that } \quad g(t) - g(t_1) = \dot{g}(t_2)\,(t - t_1)$$

and therefore, if $R_1 > 0$ is an upper bound on the function $|\dot{g}|$, one can always use $\eta = R/R_1$ independently of t_1 to verify the definition of uniform continuity.

Let us now prove Barbalat's lemma, by contradiction.

Proof of Barbalat's lemma: Assume that $\dot{f}(t)$ does not approach zero as $t \to \infty$. Then $\exists\, \varepsilon_o > 0$, $\forall\, T > 0$, $\exists\, t > T$, $|\dot{f}(t)| \geq \varepsilon_o$. Therefore, we can get an *infinite* sequence of t_i's (such that $t_i \to \infty$ as $i \to \infty$) such that $|\dot{f}(t_i)| \geq R_o$. Since $\dot{f}(t)$ is assumed to be *uniformly* continuous, $\exists\, \eta > 0$, such that for any t' and t'' satisfying $|t' - t''| < \eta$

$$|\dot{f}(t') - \dot{f}(t'')| < \frac{\varepsilon_o}{2}$$

This implies that for any t within the η-neighborhood of t_i (*i.e.*, such that $|t - t_i| < \eta$)

$$|\dot{f}(t)| > \frac{\varepsilon_o}{2}$$

Hence, for all t_i,

$$\left| \int_{t_i - \eta}^{t_i + \eta} \dot{f}(t)\, dt \right| = \int_{t_i - \eta}^{t_i + \eta} |\dot{f}|(t)\, dt \geq \frac{\varepsilon_o}{2} 2\eta = \varepsilon_o \eta$$

where the left equality comes from the fact that $\dot{f}$ keeps a constant sign over the integration interval, due to the continuity of $\dot{f}$ and the bound $|\dot{f}(t)| > \varepsilon_o/2 > 0$.

This result would contradict the known fact that the integral $\int_o^t \dot{f}(r)\, dr$ has a limit (equal to $f(\infty) - f(0)$) as $t \to \infty$. □

Given the simple sufficient condition for uniform continuity mentioned earlier, an immediate and practical corollary of Barbalat's lemma can be stated as follows: *if the differentiable function $f(t)$ has a finite limit as $t \to \infty$, and is such that $\ddot{f}$ exists and is bounded, then $\dot{f}(t) \to 0$ as $t \to \infty$.*

The following example illustrates how to assert the uniform continuity of signals in control systems.

Example 4.12: Consider a strictly stable linear system whose input is bounded. Then the system output is uniformly continuous.

Indeed, write the system in the standard form

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\,\mathbf{x}$$

Since $\mathbf{u}$ is bounded and the linear system is strictly stable, thus the state $\mathbf{x}$ is bounded. This in turn implies from the first equation that $\dot{\mathbf{x}}$ is bounded, and therefore from the second equation that

$\dot{\mathbf{y}} = \mathbf{C}\,\dot{\mathbf{x}}$ is bounded. Thus the system output $\mathbf{y}$ is uniformly continuous. □

USING BARBALAT'S LEMMA FOR STABILITY ANALYSIS

To apply Barbalat's lemma to the analysis of dynamic systems, one typically uses the following immediate corollary, which looks very much like an invariant set theorem in Lyapunov analysis:

Lemma 4.3 ("Lyapunov-Like Lemma") *If a scalar function $V(\mathbf{x}, t)$ satisfies the following conditions*

- *$V(\mathbf{x}, t)$ is lower bounded*
- *$\dot{V}(\mathbf{x}, t)$ is negative semi-definite*
- *$\dot{V}(\mathbf{x}, t)$ is uniformly continuous in time*

then $\dot{V}(\mathbf{x}, t) \to 0$ as $t \to \infty$.

Indeed, V then approaches a finite limiting value V_∞, such that $V_\infty \leq V(\mathbf{x}(0), 0)$ (this does not require uniform continuity). The above lemma then follows from Barbalat's lemma.

To illustrate this procedure, let us consider the asymptotic stability analysis of a simple adaptive control system.

Example 4.13: As we shall detail in chapter 8, the closed-loop error dynamics of an adaptive control system for a first-order plant with one unknown parameter is

$$\dot{e} = -e + \theta\, w(t)$$

$$\dot{\theta} = -\, e\, w(t)$$

where e and θ are the two states of the closed-loop dynamics, representing tracking error and parameter error, and $w(t)$ is a bounded continuous function (in the general case, the dynamics has a similar form but with e, θ, and $w(t)$ replaced by vector quantities). Let us analyze the asymptotic properties of this system.

Consider the lower bounded function

$$V = e^2 + \theta^2$$

Its derivative is

$$\dot{V} = 2e(-e + \theta w) + 2\theta(-e\, w(t)) = -2e^2 \leq 0$$

This implies that $V(t) \leq V(0)$, and therefore, that e and θ are bounded. But the invariant set theorems cannot be used to conclude the convergence of e, because the dynamics is non-autonomous.

To use Barbalat's lemma, let us check the uniform continuity of $\dot{V}$. The derivative of $\dot{V}$ is

$$\ddot{V} = -4\,e(-e+\theta w)$$

This shows that $\ddot{V}$ is bounded, since w is bounded by hypothesis, and e and θ were shown above to be bounded. Hence, $\dot{V}$ is uniformly continuous. Application of Barbalat's lemma then indicates that $e \to 0$ as $t \to \infty$.

Note that, although e converges to zero, the system *is not asymptotically stable*, because θ is only guaranteed to be bounded. □

The analysis in the above example is quite similar to a Lyapunov analysis based on invariant set theorems. Such an analysis based on Barbalat's lemma shall be called a *Lyapunov-like* analysis. It presents two subtle but important differences with Lyapunov analysis, however. The first is that the function V can simply be a lower bounded function of $\mathbf{x}$ and t instead of a positive definite function. The second difference is that the derivative $\dot{V}$ must be shown to be uniformly continuous, in addition to being negative or zero. This is typically done by proving that $\ddot{V}$ is bounded. Of course, in using the Lyapunov-like lemma for stability analysis, the primary difficulty is still the proper choice of the scalar function V.

4.6 Positive Linear Systems

In the analysis and design of nonlinear systems, it is often possible and useful to decompose the system into a linear subsystem and a nonlinear subsystem. If the transfer function (or transfer matrix) of the linear subsystem is so-called *positive real*, then it has important properties which may lead to the generation of a Lyapunov function for the whole system. In this section, we study linear systems with positive real transfer functions or transfer matrices, and their properties. Such systems, called *positive linear systems*, play a central role in the analysis and design of many nonlinear control problems, as will be seen later in the book.

4.6.1 PR and SPR Transfer Functions

We consider rational transfer functions of n^{th}-order single-input single-output linear systems, represented in the form

$$h(p) \;=\; \frac{b_m\, p^m + b_{m-1}\, p^{m-1} + \ldots + b_o}{p^n + a_{n-1}\, p^{n-1} + \ldots + a_o}$$

The coefficients of the numerator and denominator polynomials are assumed to be real numbers and $n \geq m$. The difference $n-m$ between the order of the denominator and that of the numerator is called the *relative degree* of the system.

Definition 4.10 *A transfer function $h(p)$ is positive real if*

$$Re[h(p)] \geq 0 \quad \textit{for all} \quad Re[p] \geq 0 \tag{4.33}$$

It is strictly positive real if $h(p-\varepsilon)$ is positive real for some $\varepsilon > 0$.

Condition (4.33), called the positive real condition, means that $h(p)$ always has a positive (or zero) real part when p has positive (or zero) real part. Geometrically, it means that the rational function $h(p)$ maps every point in the closed right half (*i.e.*, including the imaginary axis) of the complex plane into the closed right half of the $h(p)$ plane. The concept of positive real functions originally arose in the context of circuit theory, where the transfer function of a passive network (passive in the sense that no energy is generated in the network, *e.g.*, a network consisting of only inductors, resistors, and capacitors) is rational and positive real. In section 4.7, we shall reconcile the PR concept with passivity.

Example 4.14: A strictly positive real function

Consider the rational function

$$h(p) = \frac{1}{p+\lambda}$$

which is the transfer function of a first-order system, with $\lambda > 0$. Corresponding to the complex variable $p = \sigma + j\omega$,

$$h(p) = \frac{1}{(\sigma+\lambda)+j\omega} = \frac{\sigma+\lambda-j\omega}{(\sigma+\lambda)^2+\omega^2}$$

Obviously, $\mathrm{Re}[h(p)] \geq 0$ if $\sigma \geq 0$. Thus, $h(p)$ is a positive real function. In fact, one can easily see that $h(p)$ is strictly positive real, for example by choosing $\varepsilon = \lambda/2$ in Definition 4.9. □

For higher-order transfer functions, it is often difficult to use the definition directly in order to test the positive realness condition, because this involves checking the positivity condition over the entire right-half of the complex plane. The following theorem can simplify the algebraic complexity.

Theorem 4.10 *A transfer function $h(p)$ is strictly positive real (SPR) if and only if*

i) $h(p)$ is a strictly stable transfer function

ii) the real part of $h(p)$ is strictly positive along the $j\omega$ axis, i.e.,

$$\forall\ \omega \geq 0 \quad \text{Re}[h(j\omega)] > 0 \tag{4.34}$$

The proof of this theorem is presented in the next section, in connection with the so-called passive systems.

The above theorem implies simple *necessary* conditions for asserting whether a given transfer function $h(p)$ is SPR:

- $h(p)$ is strictly stable

- The Nyquist plot of $h(j\omega)$ lies entirely in the right half complex plane. Equivalently, the phase shift of the system in response to sinusoidal inputs is always less than 90°

- $h(p)$ has relative degree 0 or 1

- $h(p)$ is strictly minimum-phase (*i.e.*, all its zeros are strictly in the left-half plane)

The first and second conditions are immediate from the theorem. The last two conditions can be derived from the second condition simply by recalling the procedure for constructing Bode or Nyquist frequency response plots (systems with relative degree larger than 1 and non-minimum phase systems have phase shifts larger than 90^o at high frequencies, or, equivalently have parts of the Nyquist plot lying in the left-half plane).

Example 4.15: SPR and non-SPR transfer functions

Consider the following systems

$$h_1(p) = \frac{p-1}{p^2 + a\,p + b}$$

$$h_2(p) = \frac{p+1}{p^2 - p + 1}$$

$$h_3(p) = \frac{1}{p^2 + a\,p + b}$$

$$h_4(p) = \frac{p+1}{p^2 + p + 1}$$

The transfer functions h_1, h_2, and h_3 are not SPR, because h_1 is non-minimum phase, h_2 is unstable, and h_3 has relative degree larger than 1.

Is the (strictly stable, minimum-phase, and of relative degree 1) function h_4 actually SPR? We have

$$h_4(j\omega) = \frac{j\omega + 1}{-\omega^2 + j\omega + 1} = \frac{[j\omega + 1][-\omega^2 - j\omega + 1]}{[1-\omega^2]^2 + \omega^2}$$

(where the second equality is obtained by multiplying numerator and denominator by the complex conjugate of the denominator) and thus

$$\mathrm{Re}[\,h_4(j\omega)\,] = \frac{-\omega^2 + 1 + \omega^2}{[1-\omega^2]^2 + \omega^2} = \frac{1}{[1-\omega^2]^2 + \omega^2}$$

which shows that h_4 is SPR (since it is also strictly stable). Of course, condition (4.34) can also be checked directly on a computer. □

The basic difference between PR and SPR transfer functions is that PR transfer functions may tolerate poles on the $j\omega$ axis, while SPR functions cannot.

Example 4.16: Consider the transfer function of an integrator,

$$h(p) = \frac{1}{p}$$

Its value corresponding to $p = \sigma + j\omega$ is

$$h(p) = \frac{\sigma - j\omega}{\sigma^2 + \omega^2}$$

One easily sees from Definition 4.9 that $h(p)$ is PR but not SPR. □

More precisely, we have the following result, which complements Theorem 4.10.

Theorem 4.11 *A transfer function $h(p)$ is positive real if, and only if,*

i) $h(p)$ is a stable transfer function

(ii) The poles of $h(p)$ on the $j\omega$ axis are simple (i.e., distinct) and the associated residues are real and non-negative

iii) $Re[h(j\omega)] \geq 0$ for any $\omega \geq 0$ such that $j\omega$ is not a pole of $h(p)$

4.6.2 The Kalman-Yakubovich Lemma

If a transfer function of a system is SPR, there is an important mathematical property associated with its state-space representation, which is summarized in the celebrated Kalman-Yakubovich (KY) lemma.

Lemma 4.4 (Kalman-Yakubovich) *Consider a controllable linear time-invariant system*

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{b}\,u$$

$$y = \mathbf{c}^T\mathbf{x}$$

The transfer function

$$h(p) = \mathbf{c}^T[p\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b} \tag{4.35}$$

is strictly positive real if, and only if, there exist positive definite matrices **P** *and* **Q** *such that*

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \tag{4.36a}$$

$$\mathbf{P}\mathbf{b} = \mathbf{c} \tag{4.36b}$$

The proof of this lemma is presented in section 4.7 in connection with passivity in linear systems. Beyond its mathematical statement, which shall be extensively used in chapter 8 (Adaptive Control), the KY lemma has important physical interpretations and uses in generating Lyapunov functions, as discussed in section 4.7.

The KY lemma can be easily extended to PR systems. For such systems, it can be shown that there exist a positive definite matrix **P** and a positive *semi*-definite matrix **Q** such that (4.36a) and (4.36b) are verified. The usefulness of this result is that it is applicable to transfer functions containing a pure integrator ($1/p$ in the frequency-domain), of which we shall see many in chapter 8 when we study adaptive controller design. The Kalman-Yakubovich lemma is also referred to as the *positive real lemma.*

In the KY lemma, the involved system is required to be asymptotically stable and completely controllable. A modified version of the KY lemma, relaxing the controllability condition, can be stated as follows:

Lemma 4.5 (Meyer-Kalman-Yakubovich) *Given a scalar* $\gamma \geq 0$, *vectors* **b** *and* **c**, *an asymptotically stable matrix* **A**, *and a symmetric positive definite matrix* **L**, *if the transfer function*

$$H(\mathbf{p}) = \frac{\gamma}{2} + \mathbf{c}^T [p\mathbf{I} - \mathbf{A}]^{-1} \mathbf{b}$$

is SPR, then there exist a scalar $\varepsilon > 0$, *a vector* **q**, *and a symmetric positive definite matrix* **P** *such that*

$$\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A} = -\mathbf{q}\mathbf{q}^T - \varepsilon \mathbf{L}$$

$$\mathbf{P}\mathbf{b} = \mathbf{c} + \sqrt{\gamma}\, \mathbf{q}$$

This lemma is different from Lemma 4.4 in two aspects. First, the involved system now has the output equation

$$y = \mathbf{c}^T \mathbf{x} + \frac{\gamma}{2} u$$

Second, the system is only required to be stabilizable (but not necessarily controllable).

4.6.3 Positive Real Transfer Matrices

The concept of positive real transfer function can be generalized to rational positive real matrices. Such generalization is useful for the analysis and design of multi-input-multi-output nonlinear control systems.

Definition 4.11 *An* $m \times m$ *transfer matrix* $\mathbf{H}(p)$ *is called PR if*

$\mathbf{H}(p)$ *has elements which are analytic for* $\mathrm{Re}(p) > 0$;

$\mathbf{H}(p) + \mathbf{H}^T(p^*)$ *is positive semi-definite for* $\mathrm{Re}(p) > 0$.

where the asterisk * *denotes the complex conjugate transpose.* $\mathbf{H}(p)$ *is SPR if* $\mathbf{H}(p - \varepsilon)$ *is PR for some* $\varepsilon > 0$.

The Kalman-Yakubovich lemma and the Meyer-Kalman-Yakubovich lemma can be easily extended to positive real transfer matrices.

4.7 The Passivity Formalism

As we saw earlier, Lyapunov functions are generalizations of the notion of energy in a dynamic system. Thus, intuitively, we expect Lyapunov functions to be "additive", *i.e.*, Lyapunov functions for combinations of systems to be derived by simply adding the Lyapunov functions describing the subsystems. Passivity theory formalizes this intuition, and derives simple rules to describe combinations of subsystems or "blocks" expressed in a Lyapunov-like formalism. It also represents an approach to constructing Lyapunov functions or Lyapunov-like functions for feedback control purposes.

As a motivation, recall first that the dynamics of state-determined physical systems, whether linear or nonlinear, satisfy energy-conservation equations of the form

$$\frac{d}{dt}\,[\text{Stored Energy}] \;=\; [\text{External Power Input}] \;+\; [\text{Internal Power Generation}]$$

These equations actually form the basis of modeling techniques such as bond-graphs. The external power input term can be represented as the scalar product $\mathbf{y}^T\mathbf{u}$ of an input ("effort" or "flow") $\mathbf{u}$, and a output ("flow" or "effort") $\mathbf{y}$.

In the following, we shall more generally consider systems which verify equations of the form

$$\dot{V}_1(t) = \mathbf{y}_1^{\,T}\,\mathbf{u}_1 \;-\; g_1(t) \tag{4.37}$$

where $V_1(t)$ and $g_1(t)$ are scalar functions of time, $\mathbf{u}_1$ is the system input, and $\mathbf{y}_1$ is its output. Note that, from a mathematical point of view, the above form is quite general (given an arbitrary system, of input $\mathbf{u}_1(t)$ and output $\mathbf{y}_1(t)$, we can let, for instance, $g_1(t) \equiv 0$ and $V_1(t) = \int_o^t \mathbf{y}_1^{\,T}(r)\,\mathbf{u}_1(r)\,dr$). It is the physical or "Lyapunov-like" properties that $V_1(t)$ and $g_1(t)$ may have, and how they are transmitted through combinations with similar systems, that we shall be particularly interested in.

4.7.1 Block Combinations

Assume that we couple a system in the form (4.37), or *power form*, to one verifying the similar equation

$$\dot{V}_2(t) = \mathbf{y}_2^{\,T}\,\mathbf{u}_2 \;-\; g_2(t)$$

in a feedback configuration, namely $\mathbf{u}_2 = \mathbf{y}_1$ and $\mathbf{u}_1 = -\,\mathbf{y}_2$ (Figure 4.4), assuming of course that the vectors $\mathbf{u}_i$ and $\mathbf{y}_j$ are all of the same dimension. We then have

$$\frac{d}{dt}\left[\, V_1(t) + V_2(t) \,\right] = -\left[\, g_1(t) + g_2(t) \,\right] \tag{4.38}$$

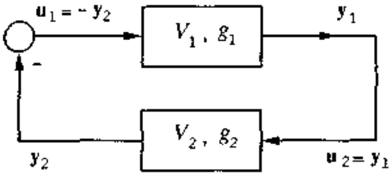


Figure 4.4 : Two blocks of the form (4.37), in a feedback configuration

Let us assume that the function $V_1 + V_2$ is lower bounded (*e.g.*, is positive). Then, using the same reasoning as in section 4.5, we have

- If $\forall\, t \geq 0\,,\ g_1(t) + g_2(t) \geq 0$, then the function $V_1 + V_2$ is upper bounded, and

$$\int_o^\infty \left[\, g_1(t) + g_2(t) \,\right] dt < \infty$$

- If in addition, the function $g_1 + g_2$ is uniformly continuous, then $\left[\, g_1(t) + g_2(t) \,\right] \rightarrow 0$ as $t \rightarrow \infty$.

- In particular, if $g_1(t)$ and $g_2(t)$ are both non-negative and uniformly continuous, then they both tend to zero as $t \rightarrow \infty$

Note that an explicit expression of $V_1 + V_2$ is not needed in the above results. More generally, without any assumption on the sign of $V_1 + V_2$ or $g_1 + g_2$, we can state that

- If $V_1 + V_2$ has a finite limit as $t \rightarrow \infty$, and if $g_1 + g_2$ is uniformly continuous, then $\left[\, g_1(t) + g_2(t) \,\right] \rightarrow 0$ as $t \rightarrow \infty$.

A system verifying an equation of the form (4.37) with V_1 lower bounded and $g_1 \geq 0$ is said to be <u>*passive*</u> (or to be a <u>*passive mapping*</u> between $\mathbf{u}_1$ and $\mathbf{y}_1$). Furthermore, a passive system is said to be <u>*dissipative*</u> if

$$\int_o^\infty \mathbf{y}_1^T(t)\,\mathbf{u}_1(t)\,dt \neq \mathbf{0} \quad \Rightarrow \quad \int_o^\infty g_1(t)\,dt > 0$$

Example 4.17: The nonlinear mass-spring-damper system

$$m\,\ddot{x} + x^2\,\dot{x}^3 + x^7 = F$$

represents a dissipative mapping from external force F to velocity $\dot{x}$, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\,\dot{x}^2 + \frac{1}{8}x^8\right) = \dot{x}\,F - x^2\,\dot{x}^4$$

Of course, here V_1 is simply the total (kinetic plus potential) energy stored in the system, and g_1 is the dissipated power. □

Example 4.18: Consider the system (Figure 4.5)

$$\dot{x} + \lambda(t)\,x = u \tag{4.39}$$

$$y = h(x)$$

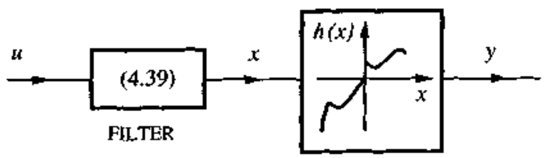


Figure 4.5 : A passive single-input single-output system

where the function h is of the same sign as its argument, although not necessarily continuous, and $\lambda(t) \geq 0$. The mapping $u \rightarrow y$ is passive, since

$$\frac{d}{dt}\int_o^x h(\xi)\,d\xi = h(x)\,\dot{x} = y\,u - \lambda(t)\,h(x)\,x$$

with $\int_o^x h(\xi)d\xi \geq 0$ and $\lambda(t)\,h(x)\,x \geq 0$ for all x. The mapping is actually dissipative if $\lambda(t)$ is not identically zero.

Of course, the function $\lambda(t)$ may actually be of the form $\lambda[x(t)]$. For instance, the system

$$\dot{x} + x^3 = u$$

$$y = x - \sin^2 x$$

is a dissipative mapping from u to y. □

A particularly convenient aspect of formalizing the construction of Lyapunov-like functions as above, is that *parallel and feedback combinations of systems in the power form are still in the power form.* Indeed, it is straightforward to verify that, for both the parallel combination and the feedback combination (Figure 4.6), one has

$$\mathbf{y}^T\mathbf{u} = \mathbf{y}_1{}^T\mathbf{u}_1 + \mathbf{y}_2{}^T\mathbf{u}_2$$

Namely, for the parallel combination

$$\mathbf{y}^T\mathbf{u} = (\mathbf{y}_1 + \mathbf{y}_2)^T\mathbf{u} = \mathbf{y}_1{}^T\mathbf{u} + \mathbf{y}_2{}^T\mathbf{u} = \mathbf{y}_1{}^T\mathbf{u}_1 + \mathbf{y}_2{}^T\mathbf{u}_2$$

and for the feedback combination

$$\mathbf{y}^T\mathbf{u} = \mathbf{y}_1{}^T(\mathbf{u}_1 + \mathbf{y}_2) = \mathbf{y}_1{}^T\mathbf{u}_1 + \mathbf{y}_1{}^T\mathbf{y}_2 = \mathbf{y}_1{}^T\mathbf{u}_1 + \mathbf{u}_2{}^T\mathbf{y}_2$$

Incidentally, this result is a particular case of what is known in circuit theory as *Tellegen's power conservation theorem.* Thus, we have, for the overall system

$$V = V_1 + V_2 \qquad\qquad g = g_1 + g_2$$

By induction, *any combination* of feedback and/or parallel combinations of systems in the power form can also be described in the power form, with the corresponding V and g simply being equal to the *sum* of the individual V_i and g_i.

$$V = \sum_i V_i \qquad\qquad g = \sum_i g_i$$

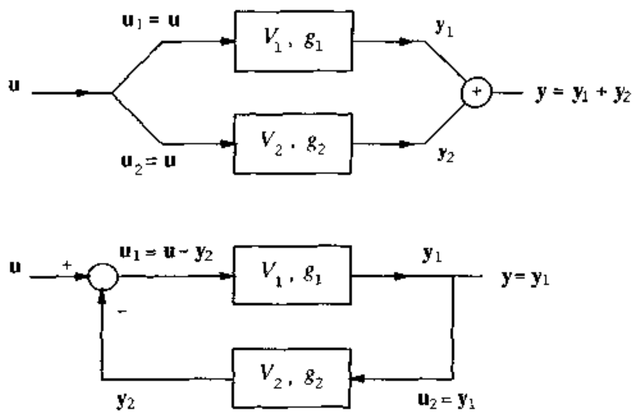


Figure 4.6 : Parallel and feedback combinations

The power of this simple result is, of course, that it does not require that the subsystems be linear.

Note that, assuming that V is lower bounded, the overall system can be passive while some of its components may be "active" ($g_i < 0$): for the system to be passive, the *sum* of the g_i has to be positive, *i.e.*, there should be more overall "power dissipation" in the system than there is "power generation". Also, note that the passivity of a block is preserved if its input or its output is multiplied by a strictly positive constant (an input gain or an output gain), since this simply multiplies the associated V_i and g_i by the same constant. Thus we have, more generally

$$V = \sum_i \alpha_i \, V_i \qquad\qquad g = \sum_i \alpha_i \; g_i$$

where α_i is the product of the input gain and the output gain for block i.

Example 4.19: Consider again the adaptive control system of Example 4.13. The fact that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}e^2 = e\,\dot{e} = e\,\theta\, w(t) - e^2$$

can be interpreted as stating that the mapping $\theta\, w(t) \rightarrow e$ is dissipative. Furthermore, using the parameter adaptation law

$$\dot{\theta} = -\, e\, w(t) \tag{4.40}$$

then corresponds to inserting a passive feedback block between e and $-\,\theta\, w(t)$, since

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\theta^2 = -\,\theta\, w(t)\, e$$ □

Note that, for a passive system, we may always assume that $g \equiv 0$, simply by adding $\int_o^t g(r)\, dr$ to the original V . Hence, the definition of passivity is often written as

$$\exists\, \alpha > -\infty\,,\ \forall\, t \geq 0\,, \qquad \int_o^t \mathbf{y}^T(r)\, \mathbf{u}(r)\, dr \;\geq\; \alpha \tag{4.41}$$

which simply says that there exists a lower bounded V such that $g \equiv 0$.

Also, note that the power form is expressed in terms of the *dot-product* $\mathbf{y}^T\mathbf{u}$. Therefore, if $\mathbf{u}_a$ and $\mathbf{y}_a$ are other choices of inputs and outputs for the system such that $\mathbf{y}_a{}^T\,\mathbf{u}_a \;=\; \mathbf{y}^T\mathbf{u}$ at all times, then they satisfy the same passivity properties as $\mathbf{u}$ and $\mathbf{y}$. For instance, if the mapping $\mathbf{u} \rightarrow \mathbf{y}$ is passive, so is the mapping $\mathbf{A}\mathbf{u} \rightarrow \mathbf{A}^{-T}\mathbf{y}$, where the matrix $\mathbf{A}$ is any (perhaps time-varying) invertible matrix. In particular, passivity is conserved through orthogonal transformations ($\mathbf{A}\,\mathbf{A}^T = \mathbf{I}$). Furthermore, note that the dimension of the vectors $\mathbf{u}_a$ and $\mathbf{y}_a$ is not necessarily the same as that of the vectors $\mathbf{u}$ and $\mathbf{y}$.

4.7.2 Passivity in Linear Systems

An important practical feature of the passivity formulation is that it is easy to characterize passive *linear* systems. This allows linear blocks to be straightforwardly incorporated or added in a nonlinear control problem formulated in terms of passive mappings.

As we shall now show, a strictly stable linear SISO system is passive if, and only if,

$$\forall\ \omega \geq 0\ ,\ \mathrm{Re}[h(j\omega)] \geq 0 \tag{4.42}$$

where h is the transfer function of the system (which we shall assume to be rational) and Re refers to the real part of a complex number. Geometrically, condition (4.42) can also be written as (Figure 4.7)

$$\forall\ \omega \geq 0\ ,\quad |\ \mathrm{Arg}\ h(j\omega)\ |\ \leq \frac{\pi}{2} \tag{4.43}$$

Thus, we see that a strictly stable linear SISO system is passive if, and only if its *phase shift* in response to a sinusoidal input is always *less than (or equal to)* 90^o.

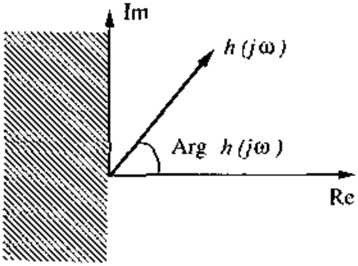


Figure 4.7 : Geometric interpretation of the passivity condition for a linear SISO system

Proof of condition (4.42): The proof is based on Parseval's theorem, which relates the time-domain and frequency-domain expressions of a signal's squared norm or "energy", as well as those of the correlation of two signals.

Consider a strictly stable linear SISO system, of transfer function $y(p)/u(p) = h(p)$, initially at rest ($y \equiv 0$) at $t = 0$. Let us apply to this system an arbitrary control input between $t = 0$ and some positive time t, and no input afterwards. Recalling expression (4.41) of passivity, we compute

$$\int_o^t y(r)\, u(r)\, dr \;=\; \int_{-\infty}^{\infty} y(r)\, u(r)\, dr \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} y(j\omega)\, u^*(j\omega)\, d\omega$$

where the first equality comes from the fact that u is zero after time t (and, of course, both u and y are zero for negative time), and the second equality comes from Parseval's theorem, with the superscript * referring to complex conjugation. Since $y(j\omega) = h(j\omega)\, u(j\omega)$, we thus have

$$\int_o^t y(r)\, u(r)\, dr \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} h(j\omega)\, |u(j\omega)|^2\, d\omega$$

Now since h is the transfer function of a real system, its coefficients are real, and thus $h(-j\omega) = [h(j\omega)]^*$. Hence,

$$\int_o^t y(r)\, u(r)\, dr \;=\; \frac{1}{\pi}\int_o^{\infty} \mathrm{Re}[h(j\omega)]\, |u(j\omega)|^2\, d\omega \tag{4.44}$$

Given expression (4.41) of passivity, equation (4.44) shows that (4.42) is a *sufficient* condition for the system to be passive. Indeed, taking an arbitrary input u, the integral $\int_o^t y(r)\, u(r)\, dr$ does not depend on the values of u at times later than t (so that our earlier assumption that u is zero after time t is not restrictive).

Equation (4.44) also shows that (4.42) is a *necessary* condition for the system to be passive. Indeed, if (4.44) was not verified, then there would be a finite interval in ω over which $\mathrm{Re}[h(j\omega)] < 0$, because h is continuous in ω. The integral could then be made arbitrarily negative by choosing $|u(j\omega)|$ large enough over this finite interval. □

Note that we have assumed that $h(p)$ is *strictly* stable, so as to guarantee the existence of the frequency-domain integrals in the above proof. Actually, using standard results in complex variable analysis, the proof can be extended easily to the case where $h(p)$ has perhaps some poles on the $j\omega$ axis, provided that these poles be simple (*i.e.*, distinct) and that the associated residues be non-negative. As discussed earlier in section 4.6, systems belonging to this more general class and verifying condition (4.42) are called *positive real (PR) systems*. Thus, a linear single-input system is passive if (and only if) it is positive real.

Condition (4.42) can also be formally stated as saying that the Nyquist plot of h is in the right half-plane. Similarly, if the Nyquist plot of a strictly stable (or PR) linear system of transfer function h is strictly in the right half-plane (except perhaps for $\omega = \infty$), *i.e*, if

$$\forall\, \omega \geq 0\,,\ \mathrm{Re}[h(j\omega)] > 0 \tag{4.45}$$

then the system is actually dissipative. As discussed in section 4.6, strictly stable linear systems verifying (4.45) are called *strictly positive real (SPR) systems*.

It can also be shown that, more generally, a strictly stable linear MIMO system is passive if, and only if

$$\forall\, \omega \geq 0\,,\ \mathbf{H}(j\omega) + \mathbf{H}^T(-j\omega)\ \geq\ \mathbf{0}$$

where $\mathbf{H}$ is the transfer matrix of the system. It is dissipative if

$$\forall\, \omega \geq 0\,,\ \mathbf{H}(j\omega) + \mathbf{H}^T(-j\omega)\ >\ \mathbf{0}$$

THE KALMAN-YAKUBOVICH LEMMA

For linear systems, the closeness of the concepts of stability and passivity can be understood easily by considering the Lyapunov equations associated with the systems, as we now show. The discussion also provides a more intuitive perspective on the KY lemma of section 4.6.2, in the light of the passivity formalism.

Recall from our discussion of Lyapunov functions for linear systems (section 3.5.1) that, for any strictly stable linear system of the form $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$, one has

$$\forall\, \mathbf{Q}\ \textit{symmetric p.d.}\,,\ \exists\, \mathbf{P}\ \textit{symmetric p.d.}\,,\ \textit{such that}\quad \mathbf{A}^T\mathbf{P} + \mathbf{P}\,\mathbf{A} = -\,\mathbf{Q} \tag{4.46}$$

an algebraic matrix equation which we referred to as the Lyapunov equation for the linear system. Letting

$$V = \frac{1}{2}\,\mathbf{x}^T\mathbf{P}\mathbf{x}$$

yields

$$\dot{V} = -\,\frac{1}{2}\,\mathbf{x}^T\mathbf{Q}\mathbf{x}$$

Consider now a linear system, *strictly stable in open-loop*, in the standard form

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} \qquad y = \mathbf{C}\mathbf{x}$$

The Lyapunov equation (4.46) is verified for this system, since it is only related to the system's stability, as characterized by the matrix $\mathbf{A}$, and is independent of the input and output matrices $\mathbf{B}$ and $\mathbf{C}$. Thus, with the same definition of V as above, $\dot{V}$ now simply contains an extra term associated with the input $\mathbf{u}$

$$\dot{V} = \mathbf{x}^T \mathbf{P}\,(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) \;=\; \mathbf{x}^T\mathbf{P}\mathbf{B}\mathbf{u} - \frac{1}{2}\,\mathbf{x}^T\mathbf{Q}\mathbf{x} \tag{4.47}$$

Since $\mathbf{y} = \mathbf{C}\mathbf{x}$, we see that (4.47) defines a dissipative mapping between $\mathbf{u}$ and $\mathbf{y}$, provided that the matrices $\mathbf{B}$ and $\mathbf{C}$ be related by

$$\mathbf{C} = \mathbf{B}^T\,\mathbf{P}$$

This result, known as the *Kalman-Yakubovich (KY) lemma*, shows the closeness of the passivity and stability concepts, given compatible choices of inputs and outputs. Since the Lyapunov equation (4.46) can be satisfied for any arbitrary symmetric *p.d.* matrix $\mathbf{Q}$, the KY lemma states that given any open-loop strictly stable linear system, one can construct an infinity of dissipative input-output maps simply by using compatible choices of inputs and outputs. In particular, given the system's physical inputs and the associated matrix $\mathbf{B}$, one can choose an infinity of outputs from which the linear system will look dissipative.

In the single-input case, and given our earlier discussion of frequency-domain characterizations of the passivity of linear systems, the KY lemma can be equivalently stated as Lemma 4.4. Note that the controllability condition in that frequency-domain formulation simply ensures that the transfer function $h(p)$ completely characterizes the linear system defined by ($\mathbf{A}$, $\mathbf{b}$, c) (since $\mathbf{P}$ is symmetric positive definite and ($\mathbf{A}$, $\mathbf{b}$) is controllable, thus $(\mathbf{A}, \mathbf{c}^T) = (\mathbf{A}, \mathbf{b}^T\mathbf{P})$ is observable). Also, as noted earlier, the KY lemma can be extended to PR systems, for which it can be shown that there exist a positive definite matrix $\mathbf{P}$ and a positive *semi*-definite matrix $\mathbf{Q}$ such that (4.36a) and (4.36b) are verified. The main usefulness of this result is that it is applicable to transfer functions containing a pure integrator, which are common in adaptive controller design.

Example 4.20: The passivity of the adaptation law (4.40) of Example 4.19 can also be shown directly by noticing that the integrator structure

$$\dot{\theta} = -\,e\,w(t)$$

implies that the mapping $-\,e\,w(t) \;\rightarrow\; \theta$ is passive, and therefore that the mapping $e \;\rightarrow -\,w(t)\,\theta$ is also passive (since $\theta\,[-\,e\,w(t)\,] = [-\,w(t)\,\theta\,]\,e$).

Furthermore, the passivity interpretation shows that the integrator in the above update law can be replaced by *any PR transfer function*, while still guaranteeing that the tracking error e tends to zero. Indeed, since the dissipation term g_2 is simply zero using the original update law, the KY lemma shows that, with the modified update law, there exists a symmetric positive definite matrix $\mathbf{P}$ and a symmetric positive *semi*-definite matrix $\mathbf{Q}$ (which, in this simple first-order case, are simply scalars $P > 0$ and $Q \geq 0$) such that

$$V = e^2 + P\,\theta^2 + \int_o^t Q\,[\theta(r)]^2\,dr$$

$$\dot{V} = -2\,e^2$$

The tracking convergence proof can then be completed as before using Barbalat's lemma.

Thus, the passivity interpretation can quickly suggest additional design flexibility. □

PASSIVITY INTERPRETATION OF SPR SYSTEMS

The passivity interpretation of SPR systems may allow one to quickly determine whether a transfer function is SPR, using physical insights.

Consider for instance the transfer function

$$h_5(p) = \frac{10p}{4p^2 + 5p + 1}$$

We can determine whether h_5 is SPR using a procedure similar to that used for the function h_4 in Example 4.15. We can also simply notice that h_5 can be *interpreted* as the transfer function of a mass-spring-damper system

$$4\ddot{x} + 5\dot{x} + x = 10u$$

$$y = \dot{x}$$

with force as input and velocity as output. Thus h_5 is dissipative, and thus SPR (since it is also strictly stable).

Finally, one can easily verify that

- If $h(p)$ is SPR, so is $1/h(p)$
- If $h_1(p)$ and $h_2(p)$ are SPR, so is

  $$h(p) = \alpha_1 h_1(p) + \alpha_2 h_2(p)$$

  provided that $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$
- If $h_1(p)$ and $h_2(p)$ are SPR, so is

  $$h(p) = \frac{h_1(p)}{1 + h_1(p)\,h_2(p)}$$

  which is the overall transfer function of the negative feedback system having $h_1(p)$ as the forward path transfer function and $h_2(p)$ as the feedback path transfer function

While these results can be derived directly, they are simply versions specific to the *linear single-input* case of more general properties of passive systems. The first result simply reflects the fact that the input u and the output y play a symmetric role in the definition of passivity. The last two results represent the linear single-input frequency-domain interpretation of our earlier general discussion on the combination of passive blocks. Actually, if either $h_1(p)$ *or* $h_2(p)$ is SPR, while the other is merely passive, then $h(p)$ is SPR. This allows us to easily construct new SPR transfer functions simply by taking any stable transfer function having a phase shift smaller than 90^o at all frequencies, and putting it in feedback or in parallel configuration with any SPR transfer function.

4.8 * Absolute Stability

The systems considered in this section have the interesting structure shown in Figure 4.8. The forward path is a linear time-invariant system, and the feedback part is a memoryless nonlinearity, *i.e.*, a nonlinear static mapping. The equations of such systems can be written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{b}\phi(y) \tag{4.48a}$$

$$y = \mathbf{c}^T\mathbf{x} \tag{4.48b}$$

where ϕ is some nonlinear function and $G(p) = \mathbf{c}^T[p\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b}$. Many systems of practical interest can be represented in this structure.

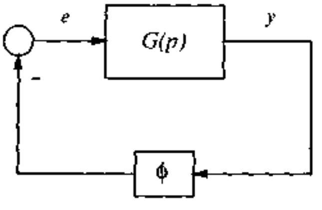


Figure 4.8 : System structure in absolute stability problems

THE ISSUE OF ABSOLUTE STABILITY

The nonlinear system in Figure 4.8 has a special structure. If the feedback path simply contains a constant gain, *i.e.*, if $\phi(y) = \alpha\, y$, then the stability of the whole system, a linear feedback system, can be simply determined by examining the eigenvalues of the closed-loop system matrix $\mathbf{A} - \alpha\mathbf{b}\mathbf{c}^T$. However, the stability analysis of the whole system with an arbitrary nonlinear feedback function ϕ is much more difficult.

In analyzing this kind of system using Lyapunov's direct method, we usually require the nonlinearity to satisfy a so-called sector condition, whose definition is given below.

Definition 4.12 *A continuous function ϕ is said to belong to the* sector *$[k_1, k_2]$, if there exists two non-negative numbers k_1 and k_2 such that*

$$y \neq 0 \quad => \quad k_1 \leq \frac{\phi(y)}{y} \leq k_2 \tag{4.49}$$

Geometrically, condition (4.49) implies that the nonlinear function always lies between the two straight lines $k_1 y$ and $k_2 y$, as shown in Figure 4.9. Two properties are implied by equation (4.49). First, it implies that $\phi(0) = 0$. Secondly, it implies that $y\phi(y) \geq 0$, *i.e*, that the graph of $\phi(y)$ lies in the first and third quadrants. Note that in many of later discussions, we will consider the special case of $\phi(y)$ belonging to the sector $[0, k]$, *i.e*, $\exists\, k > 0$, such that

$$0 \leq \phi(y) \leq k\, y \tag{4.50}$$

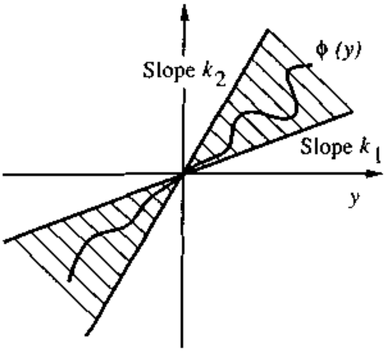


Figure 4.9 : The sector condition (4.49)

Assume that the nonlinearity $\phi(y)$ is a function belonging to the sector $[k_1, k_2]$, and that the **A** matrix of the linear subsystem in the forward path is stable (*i.e.*, Hurwitz). What additional constraints are needed to guarantee the stability of the whole system? In view of the fact that the nonlinearity in Figure 4.9 is bounded by the two straight lines, which correspond to constant gain feedback, it may be plausible that the stability of the nonlinear system should have some relation to the stability of constant gain feedback systems. In 1949, the Soviet scientist M.A. Aizerman made the following conjecture: *if the matrix $[\mathbf{A} - \mathbf{b}\mathbf{c}^T k]$ is stable for all values of k in $[k_1, k_2]$, then the nonlinear system is globally asymptotically stable.*

POPOV'S CRITERION

Aizerman's is a very interesting conjecture. If it were true, it would allow us to deduce the stability of a nonlinear system by simply studying the stability of linear systems. However, several counter-examples show that this conjecture is false. After Aizerman, many researchers continued to seek conditions that guarantee the stability of the nonlinear system in Figure 4.8. Popov's criterion imposes additional conditions on the linear subsystem, leading to a sufficient condition for asymptotic stability reminiscent of Nyquist's criterion (a necessary *and* sufficient condition) in linear system analysis.

A number of versions have been developed for Popov's criterion. The following basic version is fairly simple and useful.

Theorem 4.12 (Popov's Criterion) *If the system described by (4.48) satisfies the conditions*

- *the matrix* **A** *is Hurwitz (i.e., has all its eigenvalues strictly in the left half-plane) and the pair* [**A**, **b**] *is controllable*
- *the nonlinearity* ϕ *belongs to the sector* $[0, k]$
- *there exists a strictly positive number* α *such that*

$$\forall \omega \geq 0 \qquad \mathrm{Re}[(1 + j\alpha\omega)\, G(j\omega)] + \frac{1}{k} \geq \varepsilon \tag{4.51}$$

for an arbitrarily small $\varepsilon > 0$, *then the point* **0** *is globally asymptotically stable.*

Inequality (4.51) is called *Popov's inequality*. The criterion can be proven constructing a Lyapunov function candidate based on the KY lemma.

Let us note the main features of Popov's criterion:

- It only applies to autonomous systems.
- It is restricted to a single memoryless nonlinearity.
- The stability of the nonlinear system may be determined by examining the frequency-response functions of a linear subsystem, without the need of searching for explicit Lyapunov functions.
- It only gives a *sufficient* condition.

The criterion is most easy to apply by using its graphical interpretation. Let

$$G(j\omega) = G_1(\omega) + jG_2(\omega)$$

Then expression (4.51) can be written

$$G_1(\omega) - \alpha\omega G_2(\omega) + \frac{1}{k} \geq \varepsilon \tag{4.52}$$

Now let us construct an *associated transfer function* $\mathbf{W}(j\omega)$, with the same real part as $G(j\,\omega)$, but an imaginary part equal to $\omega\,\mathrm{Im}(G(j\,\omega))$, *i.e.*,

$$\mathbf{W}(j\omega) = x + j\,y = G_1(\omega) + j\omega G_2(\omega)$$

Then (4.52) implies that the nonlinear system is guaranteed to be globally asymptotically stable if, in the complex plane having x and y as coordinates, the polar plot of $W(j\omega)$ is (uniformly) below the line $x - \alpha y + (1/k) = 0$ (Figure 4.10). The polar plot of W is called a *Popov plot*. One easily sees the similarity of this criterion to the Nyquist criterion for linear systems. In the Nyquist criterion, the stability of a linear feedback system is determined by examining the position of the polar plot of $G(j\omega)$ relative to the point $(0, -1)$, while in the Popov criterion, the stability of a nonlinear feedback system is determined by checking the position of the associated transfer function $W(j\omega)$ with respect to a line.

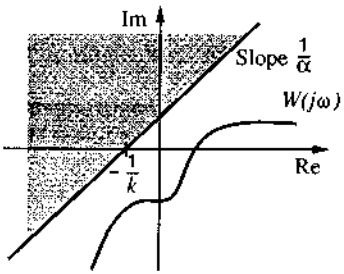


Figure 4.10 : The Popov plot

Example 4.21: Let us determine the stability of a nonlinear system of the form (4.48) where the linear subsystem is defined by

$$G(j\omega) = \frac{p+3}{p^2 + 7p + 10}$$

and the nonlinearity satisfies condition (4.50).

First, the linear subsystem is strictly stable, because its poles are -2 and -5. It is also controllable, because there is no pole-zero cancellation. Let us now check the Popov inequality. The frequency response function $G(j\omega)$ is

$$G(j\omega) = \frac{j\omega + 3}{-\omega^2 + 7\omega j + 10}$$

Therefore,

$$G_1 = \frac{4\omega^2 + 30}{\omega^4 + 29\omega^2 + 100}$$

$$G_2 = \frac{-\omega(\omega^2 + 11)}{\omega^4 + 29\omega^2 + 100}$$

Substituting the above into (4.52) leads to

$$4\omega^2 + 30 + \alpha\omega^2(\omega^2 + 11) + (\frac{1}{k} - \varepsilon)(\omega^4 + 29\omega^2 + 100) > 0$$

It is clear that this inequality can be satisfied by any strictly positive number α, and any strictly positive number k, *i.e.*, $0 < k < \infty$. Thus, the nonlinear system is globally asymptotically stable as long as the nonlinearity belongs to the first and third quadrants. □

THE CIRCLE CRITERION

A more direct generalization of Nyquist's criterion to nonlinear systems is the circle criterion, whose basic version can be stated as follows.

Theorem 4.13 (Circle Criterion) *If the system (4.48) satisfies the conditions*

- *the matrix* **A** *has no eigenvalue on the* $j\omega$ *axis, and has* ρ *eigenvalues strictly in the right half-plane;*
- *the nonlinearity* ϕ *belongs to the sector* $[k_1, k_2]$ *;*
- *one of the following is true*
    - $0 < k_1 \le k_2$, *the Nyquist plot of* $G(j\omega)$ *does not enter the disk* $\mathbf{D}(k_1, k_2)$ *and encircles it* ρ *times counter-clockwise;*
    - $0 = k_1 < k_2$, *and the Nyquist plot of* $G(j\omega)$ *stays in the half-plane* $\mathrm{Re}\, p > -1/k_2$ *;*
    - $k_1 < 0 < k_2$, *and the Nyquist plot of* $G(j\omega)$ *stays in the interior of the disk* $\mathbf{D}(k_1, k_2)$ *;*
    - $k_1 < k_2 < 0$, *the Nyquist plot of* $-G(j\omega)$ *does not enter the disk* $\mathbf{D}(-k_1, -k_2)$ *and encircles it* ρ *times counter-clockwise;*

then the equilibrium point **0** *of the system is globally asymptotically stable.*

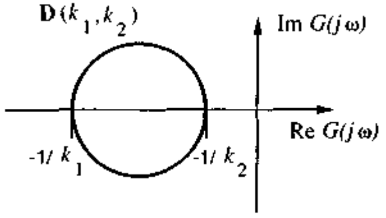


Figure 4.11 : The circle criterion

Thus we see that, essentially, the critical point $-1/k$ in Nyquist's criterion is replaced in the circle criterion by the circle of Figure 4.11 (which tends towards the point $-1/k_1$ as k_2 tends to k_1, *i.e.*, as the conic sector gets thinner). Of course, the circle criterion states *sufficient* but not necessary conditions.

The circle criterion can be extended to non-autonomous systems.

4.9 * Establishing Boundedness of Signals

In the stability analysis or convergence analysis of nonlinear systems, a frequently encountered problem is that of establishing the boundedness of certain signals. For instance, in order to use Barbalat's lemma, one has to show the uniform continuity of $\dot{f}$, which can be most conveniently shown by proving the boundedness of $\ddot{f}$. Similarly, in studying the effects of disturbances, it is also desirable to prove the boundedness of system signals in the presence of disturbances. In this section, we provide two useful results for such purposes.

THE BELLMAN-GRONWALL LEMMA

In system analysis, one can often manipulate the signal relations into an integral inequality of the form

$$y(t) \leq \int_o^t a(\tau)\, y(\tau)\, d\tau + b(t) \tag{4.53}$$

where $y(t)$, the variable of concern, appears on both sides of the inequality. The problem is to gain an explicit bound on the magnitude of y from the above inequality. The Bellman-Gronwall lemma can be used for this purpose.

Lemma 4.6 (Bellman-Gronwall) *Let a variable $y(t)$ satisfy (4.53), with $a(t)$, $b(t)$ being known real functions. Then*

$$y(t) \leq \int_o^t a(\tau)\, b(\tau) \exp[\int_\tau^t a(r)\, dr]\, d\tau + b(t) \tag{4.54}$$

If $b(t)$ is differentiable, then

$$y(t) \leq b(0) \exp[\int_o^t a(\tau)\, d\tau] + \int_o^t \dot{b}(\tau) \exp[\int_\tau^t a(r)\, dr]\, d\tau \tag{4.55}$$

In particular, if $b(t)$ is a constant, we simply have

$$y(t) \leq b(0) \exp[\int_o^t a(\tau)\, d\tau] \tag{4.56}$$

Proof: The proof is based on defining a new variable and transforming the integral inequality into a differential equation, which can be easily solved. Let

$$v(t) = \int_o^t a(\tau)\, y(\tau)\, d\tau \tag{4.57}$$

Then differentiation of v and use of (4.53) leads to

$$\dot{v} = a(t)\, y(t) \leq a(t)\, v(t) + a(t)\, b(t)$$

Let

$$s(t) = a(t)\, y(t) - a(t)\, v(t) - a(t)\, b(t)$$

which is obviously a non-positive function. Then $v(t)$ satisfies

$$\dot{v}(t) - a(t)\, v(t) = a(t)\, b(t) + s(t)$$

Solving this equation with initial condition $v(0) = 0$, yields

$$v(t) = \int_0^t \exp[\int_\tau^t a(r)\, dr]\, [a(\tau)\, b(\tau) + s(\tau)]\, d\tau \tag{4.58}$$

Since $s(\cdot)$ is a non-positive function,

$$v(t) \leq \int_o^t \exp[\int_\tau^t a(r)\, dr]\, a(\tau)\, b(\tau)\, d\tau$$

This, together with the definition of v and the inequality (4.53), leads to

$$y(t) \le \int_o^t \exp[\int_\tau^t a(r)\,dr]\, a(\tau) b(\tau) d\tau + b(t)$$

If $b(t)$ is differentiable, we obtain, by partial integration

$$\int_o^t \exp[\int_\tau^t a(r)dr]\, a(\tau)b(\tau)d\tau \;=\; -\; b(\tau)\exp[\int_\tau^t a(r)dr] \Big|_{\tau=o}^{\tau=t} + \int_o^t \dot{b}(\tau)\exp[\int_\tau^t a(r)dr]d\tau \qquad \square$$

TOTAL STABILITY

Consider the nonlinear system

$$\ddot{x} + 2\dot{x}^3 + 3x = d(t) \tag{4.59}$$

which represents a non-linear mass-spring-damper system with a disturbance $d(t)$ (which may be due to unmodeled Coulomb friction, motor ripple, parameter variations, *etc*). Is x bounded when the disturbance is bounded? This is the main question addressed by the so-called total stability theory (or stability under persistent disturbances).

In total stability, the systems concerned are described in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{g}(\mathbf{x}, t) \tag{4.60}$$

where $\mathbf{g}(\mathbf{x},t)$ is a perturbation term. Our objective is to derive a boundedness condition for the perturbed equation (4.60) from the stability properties of the associated unperturbed system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{4.61}$$

We assume that $\mathbf{x} = 0$ is an equilibrium point for the unperturbed dynamics (4.61), *i.e.*, $\mathbf{f}(\mathbf{0}, t) = 0$. But the origin is not necessarily an equilibrium point for the perturbed dynamics (4.60). The concept of total stability characterizes the ability of a system to withstand small persistent disturbances, and is defined as follows:

Definition 4.13 *The equilibrium point* $\mathbf{x} = 0$ *for the unperturbed system (4.61) is said to be* <u>*totally stable*</u> *if for every* $\varepsilon \ge 0$, *two numbers* δ_1 *and* δ_2 *exist such that* $\|\mathbf{x}(t_o)\| < \delta_1$ *and* $\|\mathbf{g}(\mathbf{x},t)\| < \delta_2$ *imply that every solution* $\mathbf{x}(t)$ *of the perturbed system (4.60) satisfies the condition* $\|\mathbf{x}(t)\| < \varepsilon$.

The above definition means that an equilibrium point is totally stable if the state of the perturbed system can be kept arbitrarily close to zero by restricting the initial state and the perturbation to be sufficiently small. Note that total stability is simply a local version (with small input) of BIBO (bounded input bounded output) stability. It is also useful to remark that if the unperturbed system is linear, then total stability is guaranteed by the asymptotic stability of the unperturbed system.

The following theorem is very useful to assert the total stability of a nonlinear system.

Theorem 4.14 *If the equilibrium point of (4.61) is uniformly asymptotically stable, then it is totally stable.*

This theorem can be proven by using the converse Lyapunov theorem 4.8. It means that uniformly asymptotically stable systems can withstand small disturbances. Because uniformly asymptotic stability can be asserted by the Lyapunov theorem 4.1, total stability of a system may be similarly established by theorem 4.1. Note that asymptotic stability is not sufficient to guarantee the total stability of a nonlinear system as can be verified by counter-examples. We also point out that exponentially stable systems are always totally stable because exponential stability implies uniform asymptotic stability.

Example 4.22: Consider again the system (4.59). Let us analyze the stability of the unperturbed system

$$\ddot{x} + 2\,\dot{x}^3 + 3x = 0$$

first. Using the scalar function

$$V(\mathbf{x}) = \frac{1}{2}\,[\,\dot{x}^2 + 3x^2]$$

and the invariant set theorem, one easily shows that the equilibrium point is globally asymptotically stable. Because the system is autonomous, the stability is also uniform. Thus, the above theorem shows that the system can withstand small disturbances $d(t)$. □

Total stability guarantees boundedness to only small-disturbance, and requires only local uniform asymptotic stability of the equilibrium point. One might wonder whether the global uniform asymptotic stability can guarantee the boundedness of the state in the presence of large (though still bounded) perturbations. The following counter-example demonstrates that this is not true.

Example 4.23: The nonlinear equation

$$\ddot{x} + f(\dot{x}) + x = w(t) \tag{4.62}$$

can be regarded as representing mass-spring-damper system containing nonlinear damping $f(\dot{x})$ and excitation force $w(t)$, where f is a first and third quadrant continuous nonlinear function such that

$$|f(y)| \leq 1 \qquad -\infty < y < \infty$$

as illustrated in Figure 4.12.

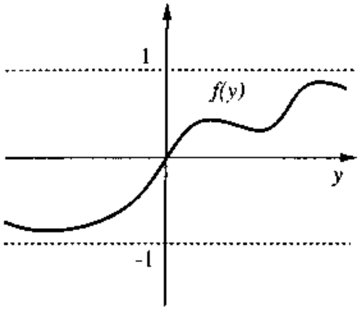


Figure 4.12 : A nonlinear damping function

The system is totally stable, because the equilibrium point can be shown to be globally uniformly asymptotically stable using the Lyapunov function $V = (1/2)(\dot{x}^2 + x^2)$. Is the output bounded for bounded input?

Let us consider the response of the system to the excitation force $w(t) = A\sin t$, $A > 8/\pi$. By writing (4.62) as

$$\ddot{x} + x = A\sin t - f(\dot{x})$$

and solving this linear equation with $(A\sin t - f(\dot{x}))$ as input, we obtain

$$x(t) = \frac{A}{2}(\sin t - t\cos t) - \int_0^t \sin(t-\tau) f(\dot{x})\, d\tau \;\geq\; \frac{A}{2}(\sin t - t\cos t) - \int_0^t |\sin(t-\tau)|\, d\tau$$

The integral on the right-hand side can be shown to be smaller than $(2/\pi)(1+\varepsilon)t$ for any $t \geq t_o$, and $\varepsilon > 0$ and some t_o. At $t_n = (2n+1)\pi$,

$$x(t_n) \geq (2n+1)\pi\,[\,\frac{A}{2} - \frac{2}{\pi}(1+\varepsilon)\,]$$

Therefore, if we take $A > 8/\pi$ and $\varepsilon = 1/2$, $x(t_n) \to \infty$. □

4.10 * Existence and Unicity of Solutions

This section discusses the mathematically important question of existence and unicity of solutions of nonlinear differential equations. We first describe a simple and quite general sufficient condition for a nonlinear differential equation to admit a solution, and then simple but conservative conditions for this solution to be unique.

Theorem 4.15 (Cauchy existence theorem) *Consider the differential equation* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, *with initial condition* $\mathbf{x}(t_o) = \mathbf{x}_o$. *If the function* $\mathbf{f}$ *is continuous in the closed region*

$$|t - t_o| \leq T\,,\ \|\mathbf{x} - \mathbf{x}_o\| \leq R$$

where T *and* R *are strictly positive constants, then the equation has at least one solution* $\mathbf{x}(t)$ *which satisfies the initial condition and is continuous over a finite time period* $[t_o\,, t_1]$ *(where* $t_1 > t_o$*).*

The above theorem indicates that the continuity of $\mathbf{f}$ is sufficient for the local *existence* of solutions. However, it does not guarantee the *uniqueness* of the solution.

Example 4.24: An equation with multiple solutions

Consider the equation

$$\dot{y} = 3\,y^{2/3}$$

with initial condition $y(0) = 0$. Two of its solutions are $y(t) \equiv 0$ and $y = t^3$. □

The following theorem gives a *sufficient* condition for the unique existence of a solution.

Theorem 4.16 *If the function* $\mathbf{f}(\mathbf{x}, t)$ *is continuous in* t, *and if there exists a strictly positive constant* L *such that*

$$\|\mathbf{f}(\mathbf{x}_2, t) - \mathbf{f}(\mathbf{x}_1, t)\| \leq L\ \|\mathbf{x}_2 - \mathbf{x}_1\| \tag{4.63}$$

for all $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *in a finite neighborhood of the origin and all* t *in the interval* $[t_o, t_o + T]$ *(with* T *being a strictly positive constant), then* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ *has a unique solution* $\mathbf{x}(t)$ *for sufficiently small initial states and in a sufficiently short time interval.*

Condition (4.63) is called a *Lipschitz condition* and L is known as a *Lipschitz constant*. If (4.63) is verified, then $\mathbf{f}$ is said to be locally Lipschitz in $\mathbf{x}$. If (4.63) is verified for any time t, then $\mathbf{f}$ is said to be locally Lipschitz in $\mathbf{x}$ uniformly with respect to t. Note that the satisfaction of a Lipschitz condition implies (locally) the continuity of $\mathbf{f}$ in terms of $\mathbf{x}$, as can be easily proven from the definition of continuity. Conversely, if locally $\mathbf{f}$ has a continuous and bounded Jacobian with respect to $\mathbf{x}$, then $\mathbf{f}$ is locally Lipschitz. When (4.63) is satisfied for any $\mathbf{x}_1$ and $\mathbf{x}_2$ in the state space, $\mathbf{f}$ is said to be *globally Lipschitz*. The above theorem can then be extended to guarantee unique existence of a solution in a global sense (*i.e.*, for any initial condition and any time period).

While the condition for existence of solutions, as stated by Cauchy's theorem, is rather benign, the sufficient condition for unicity is quite strong, and, actually, overly conservative. Most results on nonlinear dynamics simply assume that $\mathbf{f}$ is smooth enough to guarantee existence and unicity of the solutions. Note that this is always the case of good physical system models (at least in classical physics).

Actually, precise mathematical results exist about the relation between the existence of a Lyapunov function for a given system and the existence and unicity of solutions (see, *e.g.*, [Yoshizawa, 1966, 1975]). From a practical point of view, these results essentially mean that the existence of a Lyapunov function to describe a system will guarantee the system's "good behavior" under some mild smoothness assumptions on the dynamics.

4.11 Summary

Some advanced topics in nonlinear control theory are presented in this chapter. Lyapunov theory for non-autonomous systems is discussed first. Its results are quite similar to those for autonomous systems, although more involved conditions are required. A major difference is that the powerful invariant-set theorem does not apply to non-autonomous system, although Barbalat's lemma can often be a simple and effective substitute. A number of instability theorems are also presented. Such theorems are useful for non-autonomous systems, or for autonomous systems whose linearizations are only marginally stable. Theorems on the existence of Lyapunov functions may be of use in constructing Lyapunov functions for systems part of which is known to have certain stability properties. The passivity formalism is also introduced, as a notationally convenient and physically motivated interpretation of Lyapunov or Lyapunov-like analysis. The chapter also includes some results for establishing the boundedness of signals in nonlinear systems.

4.12 Notes and References

A comprehensive yet readable book on Lyapunov analysis of non-autonomous systems is [Hahn, 1967], on which most of the stability definitions in this chapter are based. The definitions and results concerning positive definite and decrescent functions are based on [Hahn, 1967; Vidyasagar, 1978]. The statement and proof of Theorem 4.1 are adapted from [Kalman and Bertram, 1960]. Example 4.5 is adapted from [Vidyasagar, 1978], Example 4.6 from [Massera, 1949], and Example 4.7 from [Rouche, *et al.*, 1977]. Figure 4.1 is adapted from A.S.M.E. Journal of Basic Engineering, 1960. In section 4.2.2, the result on perturbed linear systems is from [Vidyasagar, 1978], while the result on sufficient smoothness conditions on the $\mathbf{A}(t)$ matrix is from [Middleton, 1988]. Sections 4.2.3 and 4.3 are largely adapted from [Vidyasagar, 1978], where proofs of the main results can be found. The statement of Theorem 4.9 follows that in [Bodson, 1986]. Lemma 4.2 and its proof are from [Popov, 1973]. An extensive study of absolute stability problems from a frequency-domain perspective is contained in [Narendra and Taylor, 1973], from which the definitions and theorems on positive real functions are adapted. A more recent description of positive real functions and their applications in adaptive control can be found in [Narendra and Annasswamy, 1989]. The Bellman-Gronwall lemma and its proof are adapted from [Hsu and Meyer, 1968]. The definition and theorem on total stability are based on [Hahn, 1965]. Example 4.23 is adapted from [Desoer, *et al.*, 1965].

Passivity theory (see [Popov, 1973; Desoer and Vidyasagar, 1975]) is presented in a slightly unconventional form. Passivity interpretations of adaptive control laws are discussed in [Landau, 1979]. The reader is referred to [Vidyasagar, 1978] for a detailed discussion of absolute stability. The circle criterion and its extensions to non-autonomous systems were derived by [Narendra and Goldwyn, 1964; Sandberg, 1964; Tsypkin, 1964; Zames, 1966].

Other important robustness analysis tools include singular perturbations (see, *e.g.*, [Kokotovic, *et al.*, 1986]) and averaging (see, *e.g*, [Hale, 1980; Meerkov, 1980]).

Relations between the existence of Lyapunov functions and the existence and unicity of solutions of nonlinear differential equations are discussed in [Yoshizawa, 1966, 1975].

4.13 Exercises

4.1 Show that, for a non-autonomous system, a system trajectory is generally not an invariant set.

4.2 Analyze the stability of the dynamics (corresponding to a mass sinking in a viscous liquid)

$$\dot{v} + 2a|v|v + bv = c \qquad a > 0, b > 0$$

4.3 Show that a function $V(\mathbf{x}, t)$ is radially unbounded if, and only if, there exists a class-K function ϕ such that

$$V(\mathbf{x}, t) \geq \phi(\|\mathbf{x}\|)$$

where the function ϕ satisfies

$$\lim_{\mathbf{x} \to \infty} \phi(\|\mathbf{x}\|) = \infty$$

4.4 The performance of underwater vehicles control systems is often constrained by the "unmodeled" dynamics of the thrusters. Assume that one decides to explicitly account for thruster dynamics, based on the model

$$\dot{\omega} = -\alpha_1 \omega|\omega| + \alpha_2 \tau \qquad \alpha_1 > 0, \alpha_2 > 0$$

$$u = b\,\omega|\omega| \qquad b > 0$$

where τ is the torque input to the propeller, ω is the propeller's angular velocity, and u is the actual thrust generated.

Show that, for a *constant* torque input τ_o, the steady-state thrust is proportional to τ_o (which is consistent witht the fact that thruster dynamics is often treated as "unmodeled").

Assuming that the coefficients α_i and b in the above model are known with good accuracy, design and discuss the use of a simple "open-loop" observer for u (given an arbitrary time-varying torque input τ) in the absence of measurements of ω. (Adapted from [Yoerger and Slotine, 1990].)

4.5 Discuss the similarity of the results of section 4.2.2 with Krasovskii's theorem of section 3.5.2.

4.6 Use the first instability theorem to show the instability of the vertical-up position of a pendulum.

4.7 Show explicitly why the linear time-varying system defined by (4.18) does not satisfy the sufficient condition (4.19).

4.8 Condition (4.19) on the eigenvalues of $\mathbf{A}(t) + \mathbf{A}^T(t)$ is only, of course, a *sufficient* condition. For instance, show that the linear time-varying system associated with the matrix

$$\mathbf{A}(t) = \begin{bmatrix} -1 & e^{t/2} \\ 0 & -1 \end{bmatrix}$$

is globally asymptotically stable.

4.9 Determine whether the following systems have a stable equilibrium. Indicate whether the stability is asymptotic, and whether it is global.

(*a*) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -10 & e^{3t} \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(*b*) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 2\sin t \\ 0 & -(t+1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(*c*) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

4.10 If a differentiable function f is lower bounded and decreasing ($\dot{f} \leq 0$), then it converges to a limit. However, $\dot{f}$ does not necessarily converge to zero. Derive a counter-example. (*Hint*: You may use for $-\dot{f}$ a function that peaks periodically, but whose integral is finite.)

4.11 (a) Show that if a function f is bounded and uniformly continuous, and there exists a positive definite function $F(f, t)$ such that

$$\int_0^\infty F(f(t), t)\, dt < \infty$$

then $f(t)$ tends to zero as $t \to \infty$.

(b) For a given autonomous nonlinear system, consider a Lyapunov function V in a ball $\mathbf{B}_R$, and let ϕ be a scalar, differentiable, *strictly* monotonously increasing function of its scalar argument. Show that $[\phi(V) - \phi(0)]$ is also a Lyapunov function for the system (distinguish the cases of stability and of asymptotic stability). Suggest extensions to non-autonomous systems.

4.12 Consider a scalar, lower bounded, and twice continuously differentiable function $V(t)$ such that

$$\forall\, t \geq 0\,,\ \ \dot{V}(t) \leq 0$$

Show that, for any $t \geq 0$,

$$\dot{V}(t) = 0 \quad \Rightarrow \quad \ddot{V}(t) = 0$$

Chapter 5
Describing Function Analysis

The frequency response method is a powerful tool for the analysis and design of linear control systems. It is based on describing a linear system by a complex-valued function, the frequency response, instead of a differential equation. The power of the method comes from a number of sources. First, *graphical* representations can be used to facilitate analysis and design. Second, *physical* insights can be used, because the frequency response functions have clear physical meanings. Finally, the method's complexity only increases mildly with system order. Frequency domain analysis, however, cannot be directly applied to nonlinear systems because frequency response functions cannot be defined for nonlinear systems.

Yet, for some nonlinear systems, an extended version of the frequency response method, called the *describing function method*, can be used to *approximately* analyze and predict nonlinear behavior. Even though it is only an approximation method, the desirable properties it inherits from the frequency response method, and the shortage of other systematic tools for nonlinear system analysis, make it an indispensable component of the bag of tools of practicing control engineers. The main use of describing function method is for the prediction of limit cycles in nonlinear systems, although the method has a number of other applications such as predicting subharmonics, jump phenomena, and the response of nonlinear systems to sinusoidal inputs.

This chapter presents an introduction to the describing function analysis of nonlinear systems. The basic ideas in the describing function method are presented in section 5.1. Section 5.2 discusses typical "hard nonlinearities" in control engineering, since describing functions are particularly useful for studying control systems containing such nonlinearities. Section 5.3 evaluates the describing functions for these hard nonlinearities. Section 5.4 is devoted to the description of how to use the describing function method for the prediction of limit cycles.

5.1 Describing Function Fundamentals

In this section, we start by presenting describing function analysis using a simple example, adapted from [Hsu and Meyer, 1968]. We then provide the formal definition of describing functions and some techniques for evaluating the describing functions of nonlinear elements.

5.1.1 An Example of Describing Function Analysis

The interesting and classical Van der Pol equation

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0 \tag{5.1}$$

(where α is a positive constant) has been treated by phase-plane analysis and Lyapunov analysis in the previous chapters. Let us now study it using a different technique, which shall lead us to the concept of a describing function. Specifically, let us determine whether there exists a limit cycle in this system and, if so, calculate the amplitude and frequency of the limit cycle (pretending that we have not seen the phase portrait of the Van der Pol equation in Chapter 2). To this effect, we first assume the existence of a limit cycle with undetermined amplitude and frequency, and then determine whether the system equation can indeed sustain such a solution. This is quite similar to the assumed-variable method in differential equation theory, where we first assume a solution of certain form, substitute it into the differential equation, and then attempt to determine the coefficients in the solution.

Before carrying out this procedure, let us represent the system dynamics in a block diagram form, as shown in Figure 5.1. It is seen that the feedback system in 5.1 contains a linear block and a nonlinear block, where the linear block, although unstable, has *low-pass* properties.

Now let us assume that there is a limit cycle in the system and the oscillation signal x is in the form of

$$x(t) = A\sin(\omega t)$$

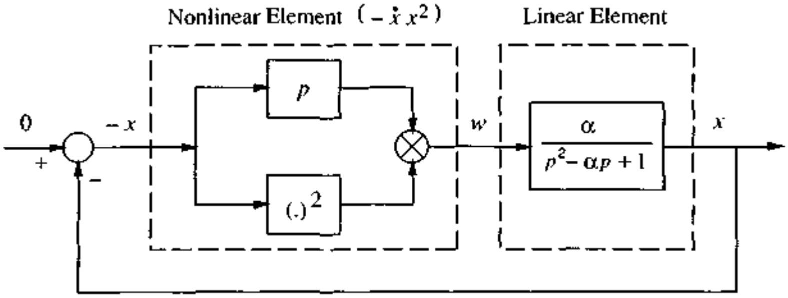


Figure 5.1 : Feedback interpretation of the Van der Pol oscillator

with A being the limit cycle amplitude and ω being the frequency. Thus,

$$\dot{x}(t) = A\,\omega\cos(\omega t)$$

Therefore, the output of the nonlinear block is

$$w = -x^2\dot{x} = -A^2\sin^2(\omega t)\;A\,\omega\cos(\omega t)$$

$$= -\frac{A^3\omega}{2}(1-\cos(2\omega t))\cos(\omega t) = -\frac{A^3\omega}{4}(\cos(\omega t)-\cos(3\omega t))$$

It is seen that w contains a third harmonic term. Since the linear block has low-pass properties, we can reasonably assume that this third harmonic term is sufficiently attenuated by the linear block and its effect is not present in the signal flow after the linear block. This means that we can approximate w by

$$w \approx -\frac{A^3}{4}\omega\cos\omega t = \frac{A^2}{4}\frac{\mathrm{d}}{\mathrm{d}t}[-A\sin(\omega t)]$$

so that the nonlinear block in Figure 5.1 can be approximated by the equivalent "quasi-linear" block in Figure 5.2. The "transfer function" of the quasi-linear block *depends on the signal amplitude* A, unlike a linear system transfer function (which is independent of the input magnitude).

In the frequency domain, this corresponds to

$$w = N(A,\omega)\,(-x) \tag{5.2}$$

where

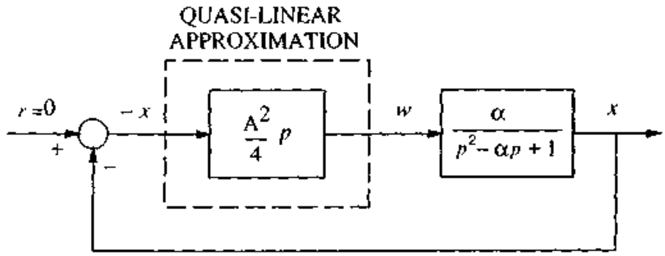


Figure 5.2 : Quasi-linear approximation of the Van der Pol oscillator

$$N(A, \omega) = \frac{A^2}{4}\,(j\omega)$$

That is, the nonlinear block can be approximated by the frequency response function $N(A, \omega)$. Since the system is assumed to contain a sinusoidal oscillation, we have

$$x = A\sin(\omega t) = G(j\omega)\,w = G(j\omega)\,N(A, \omega)\,(\,-x\,)$$

where $G(j\omega)$ is the linear component transfer function. This implies that

$$1 + \frac{A^2\,(j\omega)}{4}\,\frac{\alpha}{(j\omega)^2 - \alpha(j\omega) + 1} = 0$$

Solving this equation, we obtain

$$A = 2 \quad \omega = 1$$

Note that in terms of the Laplace variable p, the closed-loop characteristic equation of this system is

$$1 + \frac{A^2\,p}{4}\,\frac{\alpha}{p^2 - \alpha p + 1} = 0 \tag{5.3}$$

whose eigenvalues are

$$\lambda_{1,2} = -\,\frac{1}{8}\,\alpha\,(\,A^2 - 4\,)\,\pm\,\sqrt{\frac{1}{64}\,\alpha^2\,(\,A^2 - 4\,)^2\,-\,1} \tag{5.4}$$

Corresponding to $A = 2$, we obtain the eigenvalues $\lambda_{1,2} = \pm j$. This indicates the existence of a limit cycle of amplitude 2 and frequency 1. It is interesting to note neither the amplitude nor the frequency obtained above depends on the parameter α in

Equation 5.1.

In the phase plane, the above approximate analysis suggests that the limit cycle is a circle of radius 2, regardless of the value of α. To verify the plausibility of this result, the real limit cycles corresponding to the different values of α are plotted (Figure 5.3). It is seen that the above approximation is reasonable for small value of α, but that the inaccuracy grows as α increases. This is understandable because as α grows the nonlinearity becomes more significant and the quasi-linear approximation becomes less accurate.

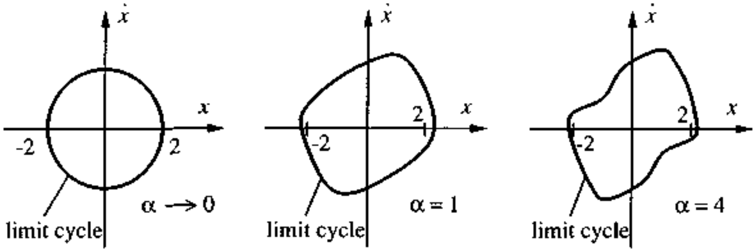


Figure 5.3 : Real limit cycles on the phase plane

The stability of the limit cycle can also be studied using the above analysis. Let us assume that the limit cycle's amplitude A is increased to a value larger than 2. Then, equation (5.4) shows that the closed-loop poles now have a negative real part. This indicates that the system becomes exponentially stable and thus the signal magnitude will decrease. Similar conclusions are obtained assuming that the limit cycle's amplitude A is decreased to a value less than 2. Thus, we conclude that the limit cycle is stable with an amplitude of 2.

Note that, in the above approximate analysis, the critical step is to replace the nonlinear block by the quasi-linear block which has the frequency response function $(A^2/4)\,(j\omega)$. Afterwards, the amplitude and frequency of the limit cycle can be determined from $1 + G(j\omega)\,N(A, \omega) = 0$. The function $N(A, \omega)$ is called the *describing function* of the nonlinear element. The above approximate analysis can be extended to predict limit cycles in other nonlinear systems which can be represented into the block diagram similar to Figure 5.1, as we shall do in section 5.4.

5.1.2 Applications Domain

Before moving on to the formal treatment of the describing function method, let us briefly discuss what kind of nonlinear systems it applies to, and what kind of information it can provide about nonlinear system behavior.

THE SYSTEMS

Simply speaking, any system which can be transformed into the configuration in Figure 5.4 can be studied using describing functions. There are at least two important classes of systems in this category.

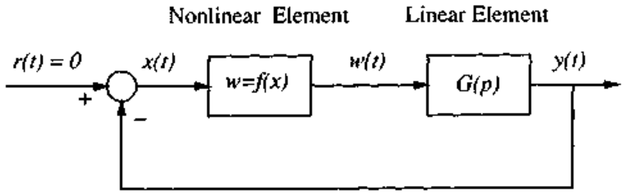


Figure 5.4 : A nonlinear system

The first important class consists of "almost" linear systems. By "almost" linear systems, we refer to systems which contain hard nonlinearities in the control loop but are otherwise linear. Such systems arise when a control system is designed using linear control but its implementation involves hard nonlinearities, such as motor saturation, actuator or sensor dead-zones, Coulomb friction, or hysteresis in the plant. An example is shown in Figure 5.5, which involves hard nonlinearities in the actuator.

Example 5.1: A system containing only one nonlinearity

Consider the control system shown in Figure 5.5. The plant is linear and the controller is also linear. However, the actuator involves a hard nonlinearity. This system can be rearranged into the form of Figure 5.4 by regarding $G_p G_1 G_2$ as the linear component G, and the actuator nonlinearity as the nonlinear element. □

"Almost" linear systems involving sensor or plant nonlinearities can be similarly rearranged into the form of Figure 5.4.

The second class of systems consists of genuinely nonlinear systems whose dynamic equations can actually be rearranged into the form of Figure 5.4. We saw an example of such systems in the previous section.

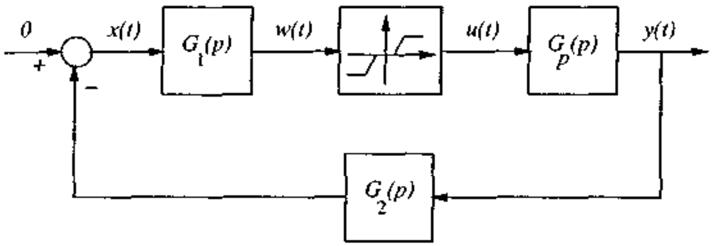


Figure 5.5 : A control system with hard nonlinearity

APPLICATIONS OF DESCRIBING FUNCTIONS

For systems such as the one in Figure 5.5, limit cycles can often occur due to the nonlinearity. However, linear control cannot predict such problems. Describing functions, on the other hand, can be conveniently used to discover the existence of limit cycles and determine their stability, regardless of whether the nonlinearity is "hard" or "soft." The applicability to limit cycle analysis is due to the fact that the form of the signals in a limit-cycling system is usually approximately *sinusoidal*. This can be conveniently explained on the system in Figure 5.4. Indeed, asssume that the linear element in Figure 5.4 has low-pass properties (which is the case of most physical systems). If there is a limit cycle in the system, then the system signals must all be periodic. Since, as a periodic signal, the input to the linear element in Figure 5.4 can be expanded as the sum of many harmonics, and since the linear element, because of its low-pass property, filters out higher frequency signals, the output $y(t)$ must be composed mostly of the lowest harmonics. Therefore, it is appropriate to assume that the signals in the whole system are basically sinusoidal in form, thus allowing the technique in subsection 5.1.1 to be applied.

Prediction of limit cycles is very important, because limit cycles can occur frequently in physical nonlinear system. Sometimes, a limit cycle can be desirable. This is the case of limit cycles in the electronic oscillators used in laboratories. Another example is the so-called dither technique which can be used to minimize the negative effects of Coulomb friction in mechanical systems. In most control systems, however, limit cycles are undesirable. This may be due to a number of reasons:

1. limit cycle, as a way of instability, tends to cause poor control accuracy
2. the constant oscillation associated with the limit cycles can cause increasing wear or even mechanical failure of the control system hardware
3. limit cycling may also cause other undesirable effects, such as passenger

discomfort in an aircraft under autopilot

In general, although a precise knowledge of the waveform of a limit cycle is usually not mandatory, the knowledge of the limit cycle's existence, as well as that of its approximate amplitude and frequency, is critical. The describing function method can be used for this purpose. It can also guide the design of compensators so as to avoid limit cycles.

5.1.3 Basic Assumptions

Consider a nonlinear system in the general form of Figure 5.4. In order to develop the *basic version* of the describing function method, the system has to satisfy the following four conditions:

1. *there is only a single nonlinear component*
2. *the nonlinear component is time-invariant*
3. *corresponding to a sinusoidal input* $x = \sin(\omega t)$, *only the fundamental component* $w_1(t)$ *in the output* $w(t)$ *has to be considered*
4. *the nonlinearity is odd*

The first assumption implies that if there are two or more nonlinear components in a system, one either has to lump them together as a single nonlinearity (as can be done with two nonlinearities in parallel), or retain only the primary nonlinearity and neglect the others.

The second assumption implies that we consider only autonomous nonlinear systems. It is satisfied by many nonlinearities in practice, such as saturation in amplifiers, backlash in gears, Coulomb friction between surfaces, and hysteresis in relays. The reason for this assumption is that the Nyquist criterion, on which the describing function method is largely based, applies only to linear time-invariant systems.

The third assumption is the *fundamental assumption* of the describing function method. It represents an *approximation*, because the output of a nonlinear element corresponding to a sinusoidal input usually contains higher harmonics besides the fundamental. This assumption implies that the higher-frequency harmonics can all be neglected in the analysis, as compared with the fundamental component. For this assumption to be valid, it is important for the linear element following the nonlinearity to have low-pass properties, *i.e.*,

$$|G(j\omega)| >> |G(jn\omega)| \quad \text{for } n = 2, 3, ... \tag{5.5}$$

This implies that higher harmonics in the output will be filtered out significantly. Thus, the third assumption is often referred to as the *filtering hypothesis*.

The fourth assumption means that the plot of the nonlinearity relation $f(x)$ between the input and output of the nonlinear element is symmetric about the origin. This assumption is introduced for simplicity, *i.e.*, so that the static term in the Fourier expansion of the output can be neglected. Note that the common nonlinearities discussed before all satisfy this assumption.

The relaxation of the above assumptions has been widely studied in literature, leading to describing function approaches for general situations, such as multiple nonlinearities, time-varying nonlinearities, or multiple-sinusoids. However, these methods based on relaxed conditions are usually much more complicated than the basic version, which corresponds to the above four assumptions. In this chapter, we shall mostly concentrate on the basic version.

5.1.4 Basic Definitions

Let us now discuss how to represent a *nonlinear component* by a describing function. Let us consider a sinusoidal input to the nonlinear element, of amplitude A and frequency ω, *i.e.*, $x(t) = A\sin(\omega t)$, as shown in Figure 5.6. The output of the nonlinear component $w(t)$ is often a periodic, though generally non-sinusoidal, function. Note that this is always the case if the nonlinearity $f(x)$ is single-valued, because the output is $f[A\sin(\omega(t+2\pi/\omega))] = f[A\sin(\omega t)]$. Using Fourier series, the periodic function $w(t)$ can be expanded as

$$w(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)] \tag{5.6}$$

where the Fourier coefficients a_i's and b_i's are generally functions of A and ω, determined by

$$a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\, d(\omega t) \tag{5.7a}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\cos(n\omega t)\, d(\omega t) \tag{5.7b}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\sin(n\omega t)\, d(\omega t) \tag{5.7c}$$

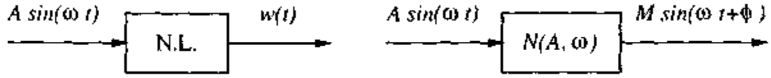


Figure 5.6 : A nonlinear element and its describing function representation

Due to the fourth assumption above, one has $a_o = 0$. Furthermore, the third assumption implies that we only need to consider the fundamental component $w_1(t)$, namely

$$w(t) \approx w_1(t) = a_1 \cos(\omega t) + b_1 \sin(\omega t) = M \sin(\omega t + \phi) \tag{5.8}$$

where

$$M(A,\omega) = \sqrt{a_1{}^2 + b_1{}^2} \quad \text{and} \quad \phi(A,\omega) = \arctan(a_1/b_1).$$

Expression (5.8) indicates that the fundamental component corresponding to a sinusoidal input is a sinusoid at the same frequency. In complex representation, this sinusoid can be written as $w_1 = M e^{j(\omega t + \phi)} = (b_1 + j a_1)\, e^{j\omega t}$.

Similarly to the concept of frequency response function, which is the frequency-domain ratio of the sinusoidal input and the sinusoidal output of a system, we define the ___describing function___ of the nonlinear element to be *the complex ratio of the fundamental component of the nonlinear element by the input sinusoid, i.e.*,

$$N(A,\omega) = \frac{M e^{j(\omega t + \phi)}}{A e^{j\omega t}} = \frac{M}{A} e^{j\phi} = \frac{1}{A}(b_1 + j a_1) \tag{5.9}$$

With a describing function representing the nonlinear component, the nonlinear element, in the presence of sinusoidal input, can be treated as if it were a linear element with a frequency response function $N(A,\omega)$, as shown in Figure 5.6. The concept of a describing function can thus be regarded as an extension of the notion of frequency response. For a linear dynamic system with frequency response function $H(j\omega)$, the describing function is independent of the input gain, as can be easily shown. However, the describing function of a nonlinear element differs from the frequency response function of a linear element in that it depends on the input amplitude A. Therefore, representing the nonlinear element as in Figure 5.6 is also called quasi-linearization.

Generally, the describing function depends on the frequency and amplitude of the input signal. There are, however, a number of special cases. When the nonlinearity is *single-valued*, the describing function $N(A,\omega)$ is *real and independent of the input frequency* ω. The realness of N is due to the fact that $a_1 = 0$, which is true because $f[A\sin(\omega t)]\cos(\omega t)$, the integrand in the expression (5.7b) for a_1, is an odd function of ωt, and the domain of integration is the symmetric interval $[-\pi, \pi]$. The frequency-independent nature is due to the fact that the integration of the single-valued function $f[A\sin(\omega t)]\sin(\omega t)$ in expression (5.7c) is done for the variable ωt, which implies that ω does not explicitly appear in the integration.

Although we have implicitly assumed the nonlinear element to be a scalar nonlinear function, the definition of the describing function also applies to the case when the nonlinear element contains dynamics (*i.e.*, is described by differential equations instead of a function). The derivation of describing functions for such nonlinear elements is usually more complicated and may require experimental evaluation.

5.1.5 Computing Describing Functions

A number of methods are available to determine the describing functions of nonlinear elements in control systems, based on definition (5.9). We now briefly describe three such methods: analytical calculation, experimental determination, and numerical integration. Convenience and cost in each particular application determine which method should be used. One thing to remember is that precision is not critical in evaluating describing functions of nonlinear elements, because the describing function method is itself an approximate method.

ANALYTICAL CALCULATION

When the nonlinear characteristics $w = f(x)$ (where x is the input and w the output) of the nonlinear element are described by an explicit function and the integration in (5.7) can be easily carried out, then analytical evaluation of the describing function based on (5.7) is desirable. The explicit function $f(x)$ of the nonlinear element may be an idealized representation of simple nonlinearities such as saturation and dead-zone, or it may be the curve-fit of an input-output relationship for the element. However, for nonlinear elements which evade convenient analytical expressions or contain dynamics, the analytical technique is difficult.

NUMERICAL INTEGRATION

For nonlinearities whose input-output relationship $w = f(x)$ is given by graphs or tables, it is convenient to use numerical integration to evaluate the describing functions. The idea is, of course, to approximate integrals in (5.7) by discrete sums over small intervals. Various numerical integration schemes can be applied for this purpose. It is obviously important that the numerical integration be easily implementable by computer programs. The result is a plot representing the describing function, which can be used to predict limit cycles based on the method to be developed in section 5.4.

EXPERIMENTAL EVALUATION

The experimental method is particularly suitable for complex nonlinearities and dynamic nonlinearities. When a system nonlinearity can be isolated and excited with sinusoidal inputs of known amplitude and frequency, experimental determination of the describing function can be obtained by using a harmonic analyzer on the output of the nonlinear element. This is quite similar to the experimental determination of frequency response functions for linear elements. The difference here is that not only the frequencies, but also the *amplitudes* of the input sinusoidal should be varied. The results of the experiments are a set of curves on complex planes representing the describing function $N(A, \omega)$, instead of analytical expressions. Specialized instruments are available which automatically compute the describing functions of nonlinear elements based on the measurement of nonlinear element response to harmonic excitation.

Let us illustrate on a simple nonlinearity how to evaluate describing functions using the analytical technique.

Example 5.2: Describing function of a hardening spring

The characteristics of a hardening spring are given by

$$w = x + x^3/2$$

with x being the input and w being the output. Given an input $x(t) = A\sin(\omega t)$, the output $w(t) = A\sin(\omega t) + A^3\sin^3(\omega t)/2$ can be expanded as a Fourier series, with the fundamental being

$$w_1(t) = a_1\cos\omega t + b_1\sin\omega t$$

Because $w(t)$ is an odd function, one has $a_1 = 0$, according to (5.7). The coefficient b_1 is

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi}[A\sin(\omega t) + A^3\sin^3(\omega t)/2]\sin(\omega t)\,d(\omega t) = A + \frac{3}{8}A^3$$

Therefore, the fundamental is

$$w_1 = (A + \frac{3}{8}A^3)\sin(\omega t)$$

and the describing function of this nonlinear component is

$$N(A,\omega) = N(A) = 1 + \frac{3}{8}A^2$$

Note that due to the odd nature of this nonlinearity, the describing function is real, being a function only of the amplitude of the sinusoidal input. □

5.2 Common Nonlinearities In Control Systems

In this section, we take a closer look at the nonlinearities found in control systems. Consider the typical system block shown in Figure 5.7. It is composed of four parts: a plant to be controlled, sensors for measurement, actuators for control action, and a control law, usually implemented on a computer. Nonlinearities may occur in any part of the system, and thus make it a nonlinear control system.

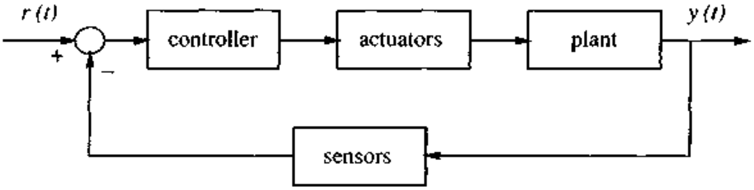


Figure 5.7 : Block diagram of control systems

CONTINUOUS AND DISCONTINUOUS NONLINEARITIES

Nonlinearities can be classified as *continuous* and *discontinuous*. Because discontinuous nonlinearities cannot be locally approximated by linear functions, they are also called "hard" nonlinearities. Hard nonlinearities are commonly found in control systems, both in small range operation and large range operation. Whether a system in small range operation should be regarded as nonlinear or linear depends on the magnitude of the hard nonlinearities and on the extent of their effects on the system performance.

Because of the common occurence of hard nonlinearities, let us briefly discuss the characteristics and effects of some important ones.

Saturation

When one increases the input to a physical device, the following phenomenon is often observed: when the input is small, its increase leads to a corresponding (often proportional) increase of output; but when the input reaches a certain level, its further increase does produces little or no increase of the output. The output simply stays around its maximum value. The device is said to be in *saturation* when this happens. Simple examples are transistor amplifiers and magnetic amplifiers. A saturation nonlinearity is usually caused by limits on component size, properties of materials, and available power. A typical saturation nonlinearity is represented in Figure 5.8, where the thick line is the real nonlinearity and the thin line is an idealized saturation nonlinearity.

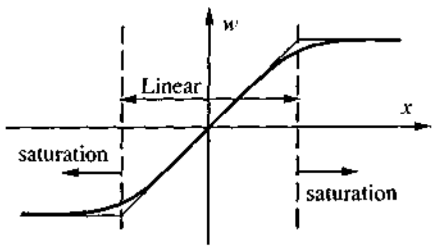


Figure 5.8 : A saturation nonlinearity

Most actuators display saturation characteristics. For example, the output torque of a two-phase servo motor cannot increase infinitely and tends to saturate, due to the properties of the magnetic material. Similarly, valve-controlled hydraulic servo motors are saturated by the maximum flow rate.

Saturation can have complicated effects on control system performance. Roughly speaking, the occurence of saturation amounts to reducing the gain of the device (*e.g.*, the amplifier) as the input signals are increased. As a result, if a system is unstable in its linear range, its divergent behavior may be suppressed into a self-sustained oscillation, due to the inhibition created by the saturating component on the system signals. On the other hand, in a linearly stable system, saturation tends to slow down the response of the system, because it reduces the effective gain.

On-off nonlinearity

An extreme case of saturation is the *on-off* or relay nonlinearity. It occurs when the linearity range is shrunken to zero and the slope in the linearity range becomes vertical. Important examples of on-off nonlinearities include output torques of gas jets for spacecraft control (as in example 2.5) and, of course, electrical relays. On-off nonlinearities have effects similar to those of saturation nonlinearities. Furthermore they can lead to "chattering" in physical systems due to their discontinuous nature.

Dead-zone

In many physical devices, the output is zero until the magnitude of the input exceeds a certain value. Such an input-output relation is called a *dead-zone*. Consider for instance a d.c. motor. In an idealistic model, we assume that any voltage applied to the armature windings will cause the armature to rotate, with small voltage causing small motion. In reality, due to the static friction at the motor shaft, rotation will occur only if the torque provided by the motor is sufficiently large. Similarly, when transmitting motion by connected mechanical components, dead zones result from manufacturing clearances. Similar dead-zone phenomena occur in valve-controlled pneumatic actuators and in hydraulic components.

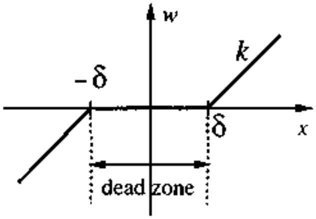


Figure 5.9 : A dead-zone nonlinearity

Dead-zones can have a number of possible effects on control systems. Their most common effect is to decrease static output accuracy. They may also lead to limit cycles or system instability because of the lack of response in the dead zone. In some cases, however, they may actually stabilize a system or suppress self-oscillations. For example, if a dead-zone is incorporated into an ideal relay, it may lead to the avoidance of the oscillation at the contact point of the relay, thus eliminating sparks and reducing wear at the contact point. In chapter 8, we describe a dead-zone technique to improve the robustness of adaptive control systems with respect to measurement noise.

Backlash and hysteresis

Backlash often occurs in transmission systems. It is caused by the small gaps which exist in transmission mechanisms. In gear trains, there always exist small gaps between a pair of mating gears, due to the unavoidable errors in manufacturing and assembly. Figure 5.10 illustrates a typical situation. As a result of the gaps, when the driving gear rotates a smaller angle than the gap b, the driven gear does not move at all, which corresponds to the dead-zone (OA segment in Figure 5.10); after contact has been established between the two gears, the driven gear follows the rotation of the driving gear in a linear fashion (AB segment). When the driving gear rotates in the reverse direction by a distance of $2b$, the driven gear again does not move, corresponding to the BC segment in Figure 5.10. After the contact between the two gears is re-established, the driven gear follows the rotation of the driving gear in the reverse direction (CD segment). Therefore, if the driving gear is in periodic motion, the driven gear will move in the fashion represented by the closed path EBCD. Note that the height of B, C, D, E in this figure depends on the amplitude of the input sinusoidal.

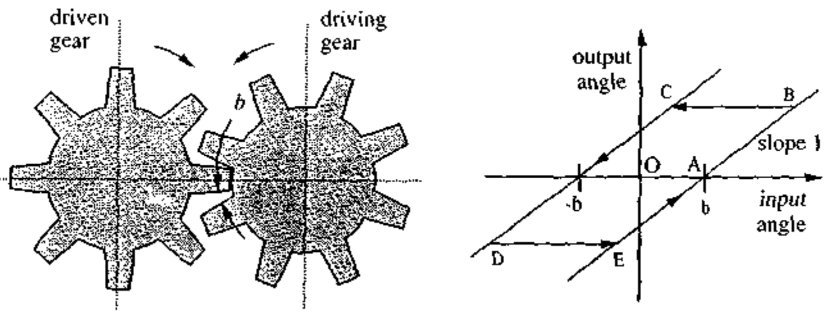


Figure 5.10 : A backlash nonlinearity

A critical feature of backlash is its multi-valued nature. Corresponding to each input, two output values are possible. Which one of the two occur depends on the history of the input. We remark that a similar multi-valued nonlinearity is hysteresis, which is frequently observed in relay components.

Multi-valued nonlinearities like backlash and hysteresis usually lead to energy storage in the system. Energy storage is a frequent cause of instability and self-sustained oscillation.

5.3 Describing Functions of Common Nonlinearities

In this section, we shall compute the describing functions for a few common nonlinearities. This will not only allow us to familiarize ourselves with the frequency domain properties of these common nonlinearities, but also will provide further examples of how to derive describing functions for nonlinear elements.

SATURATION

The input-output relationship for a saturation nonlinearity is plotted in Figure 5.11, with a and k denoting the range and slope of the linearity. Since this nonlinearity is single-valued, we expect the describing function to be a real function of the input

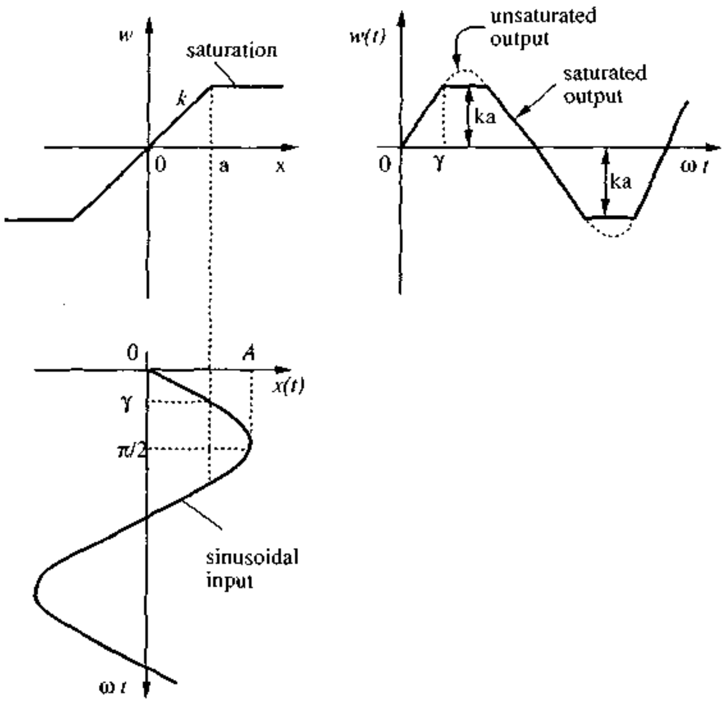


Figure 5.11 : Saturation nonlinearity and the corresponding input-output relationship

amplitude.

Consider the input $x(t) = A\sin(\omega t)$. If $A \le a$, then the input remains in the linear range, and therefore, the output is $w(t) = kA\sin(\omega t)$. Hence, the describing function is simply a constant k.

Now consider the case $A > a$. The input and the output functions are plotted in Figure 5.11. The output is seen to be symmetric over the four quarters of a period. In the first quarter, it can be expressed as

$$w(t) = \begin{cases} kA\sin(\omega t) & 0 \le \omega t \le \gamma \\ ka & \gamma < \omega t \le \pi/2 \end{cases}$$

where $\gamma = \sin^{-1}(a/A)$. The odd nature of $w(t)$ implies that $a_1 = 0$ and the symmetry over the four quarters of a period implies that

$$b_1 = \frac{4}{\pi}\int_o^{\pi/2} w(t)\sin(\omega t)\, d(\omega t)$$

$$= \frac{4}{\pi}\int_o^{\gamma} kA\sin^2(\omega t)\, d(\omega t) + \frac{4}{\pi}\int_{\gamma}^{\pi/2} ka\sin(\omega t)\, d(\omega t)$$

$$= \frac{2kA}{\pi}\left[\gamma + \frac{a}{A}\sqrt{1-\frac{a^2}{A^2}}\,\right] \tag{5.10}$$

Therefore, the describing function is

$$N(A) = \frac{b_1}{A} = \frac{2k}{\pi}\left[\sin^{-1}\frac{a}{A} + \frac{a}{A}\sqrt{1-\frac{a^2}{A^2}}\,\right] \tag{5.11}$$

The normalized describing function ($N(A)/k$) is plotted in Figure 5.12 as a function of A/a. One can observe three features for this describing function:

1. $N(A) = k$ if the input amplitude is in the linearity range
2. $N(A)$ decreases as the input amplitude increases
3. there is no phase shift

The first feature is obvious, because for small signals the saturation is not displayed. The second is intuitively reasonable, since saturation amounts to reduce the ratio of the output to input. The third is also understandable because saturation does not cause the delay of the response to input.

As a special case, one can obtain the describing function for the relay-type (on-off) nonlinearity shown in Figure 5.13. This case corresponds to shrinking the linearity range in the saturation function to zero, *i.e.*, $a \to 0$, $k \to \infty$, but $ka = M$. Though b_1 can be obtained from (5.10) by taking the limit, it is more easily obtained directly as

$$b_1 = \frac{4}{\pi}\int_0^{\pi/2} M\sin(\omega t)\, d(\omega t) = \frac{4}{\pi}M$$

Therefore, the describing function of the relay nonlinearity is

$$N(A) = \frac{4M}{\pi A} \tag{5.12}$$

The normalized describing function (N/M) is plotted in Figure 5.13 as a function of

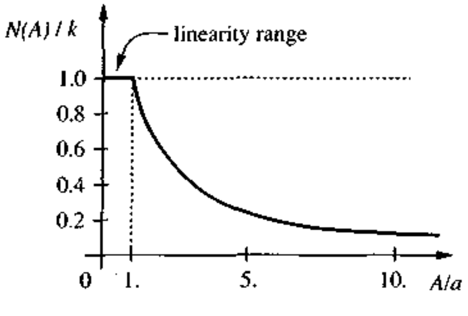


Figure 5.12 : Describing function of the saturation nonlinearity

input amplitude. Although the describing function again has no phase shift, the flat segment seen in Figure 5.12 is missing in this plot, due to the completely nonlinear nature of the relay. The asymptic properties of the describing function curve in Figure 5.13 are particularly interesting. When the input is infinitely small, the describing function is infinitely large. When the input is infinitely large, the describing function is infinitely small. One can gain an intuitive understanding of these properties by considering the ratio of the output to input for the on-off nonlinearity.

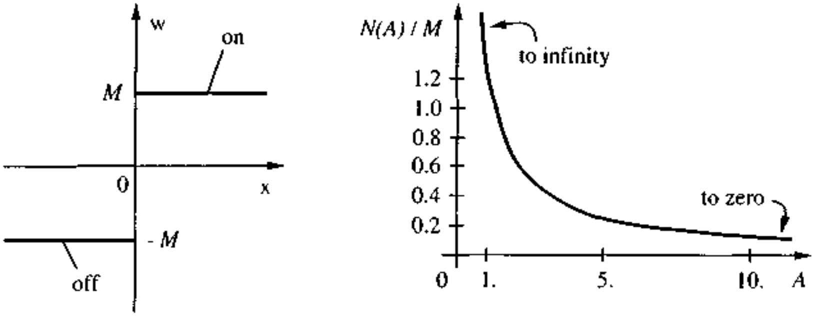


Figure 5.13 : Relay nonlinearity and its describing function

DEAD-ZONE

Consider the dead-zone characteristics shown in Figure 5.9, with the dead-zone width being 2δ and its slope k. The response corresponding to a sinusoidal input $x(t) = A\sin(\omega t)$ into a dead-zone of width 2δ and slope k, with $A \geq \delta$, is plotted in Figure 5.14. Since the characteristics is an odd function, $a_1 = 0$. The response is also seen to be symmetric over the four quarters of a period. In one quarter of a period, *i.e.*, when $0 \leq \omega t \leq \pi/2$, one has

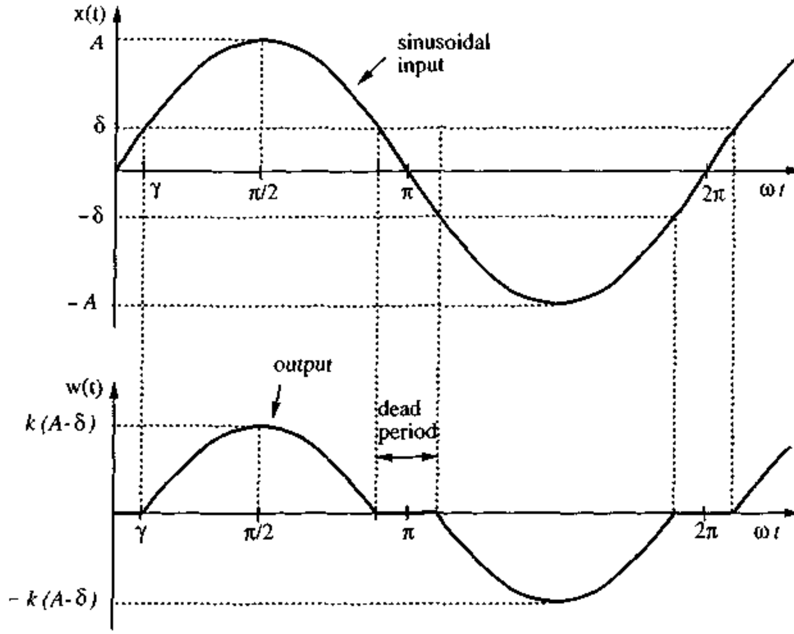


Figure 5.14 : Input and output functions for a dead-zone nonlinearity

$$w(t) = \begin{cases} 0 & 0 \le \omega t \le \gamma \\ k(A\sin(\omega t) - \delta) & \gamma \le \omega t \le \pi/2 \end{cases}$$

where $\gamma = \sin^{-1}(\delta/A)$. The coefficient b_1 can be computed as follows

$$b_1 = \frac{4}{\pi}\int_0^{\pi/2} w(t)\,\sin(\omega t)\,d(\omega t) \;=\; \frac{4}{\pi}\int_\gamma^{\pi/2} k(A\sin(\omega t) - \delta)\,\sin(\omega t)\,d(\omega t)$$

$$= \frac{2kA}{\pi}\left(\frac{\pi}{2} - \sin^{-1}\frac{\delta}{A} - \frac{\delta}{A}\sqrt{1 - \frac{\delta^2}{A^2}}\,\right) \qquad (5.13)$$

This leads to

$$N(A) = \frac{2k}{\pi}\left(\frac{\pi}{2} - \sin^{-1}\frac{\delta}{A} - \frac{\delta}{A}\sqrt{1 - \frac{\delta^2}{A^2}}\,\right)$$

This describing function $N(A)$ is a *real* function and, therefore, there is no phase shift (reflecting the absence of time-delay). The normalized describing function is plotted in Figure 5.15. It is seen that $N(A)/k$ is zero when $A/\delta < 1$, and increases up to 1 with A/δ. This increase indicates that the effect of the dead-zone gradually diminishes as the amplitude of the input signal is increased, consistently with intuition.

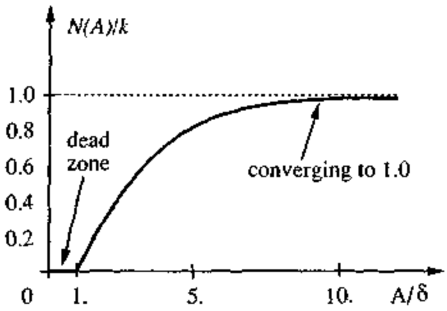


Figure 5.15 : Describing function of the dead-zone nonlinearity

BACKLASH

The evaluation of the describing functions for backlash nonlinearity is more tedious. Figure 5.16 shows a backlash nonlinearity, with slope k and width $2b$. If the input amplitude is smaller than b, there is no output. In the following, let us consider the input being $x(t) = A\sin(\omega t)$, $A \geq b$. The output $w(t)$ of the nonlinearity is as shown in the figure. In one cycle, the function $w(t)$ can be represented as

$$w(t) = (A - b)k \qquad \frac{\pi}{2} < \omega t \leq \pi - \gamma$$

$$w(t) = (A\sin(\omega t) + b)k \qquad \pi - \gamma < \omega t \leq \frac{3\pi}{2}$$

$$w(t) = -(A - b)k \qquad \frac{3\pi}{2} < \omega t \leq 2\pi - \gamma$$

$$w(t) = (A\sin(\omega t) - b)k \qquad 2\pi - \gamma < \omega t \leq \frac{5\pi}{2}$$

where $\gamma = \sin^{-1}(1 - 2b/A)$.

Unlike the previous nonlinearities, the function $w(t)$ here is neither odd nor even. Therefore, a_1 and b_1 are both nonzero. Using (5.7b) and (5.7c), we find through some tedious integrations that

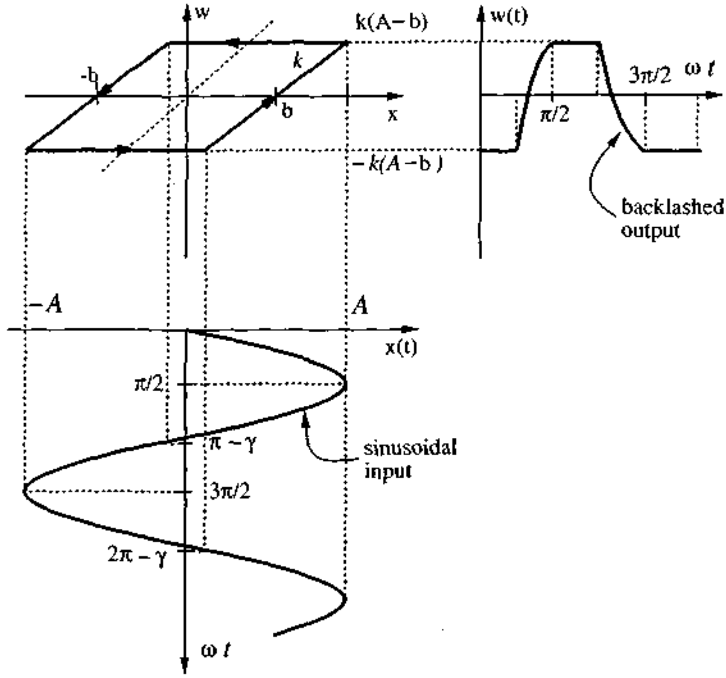


Figure 5.16 : Input and output functions for a backlash nonlinearity

$$a_1 = \frac{4kb}{\pi}\,(\frac{b}{A} - 1)$$

$$b_1 = \frac{Ak}{\pi}\,[\,\frac{\pi}{2} - \sin^{-1}(\frac{2b}{A} - 1) - (\frac{2b}{A} - 1)\,\sqrt{1 - (\frac{2b}{A} - 1)^2}$$

Therefore, the describing function of the backlash is given by

$$|N(A)| = \frac{1}{A}\sqrt{{a_1}^2 + {b_1}^2} \tag{5.14a}$$

$$\underline{/N(A)} = \tan^{-1}(a_1/b_1) \tag{5.14b}$$

The amplitude of the describing function for backlash is plotted in Figure 5.17.

We note a few interesting points :

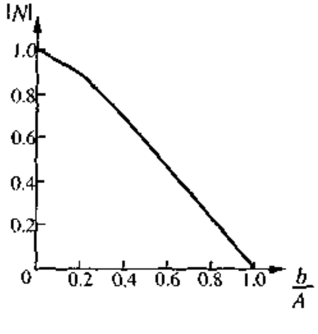


Figure 5.17 : Amplitude of describing function for backlash

1. $|N(A)| = 0 \quad if\ A = b$.
2. $|N(A)|$ increases, when b/A decreases.
3. $|N(A)| \to 1 \quad as\ b/A \to 0$.

The phase angle of the describing function is plotted in Figure 5.18. Note that a phase lag (up to 90^o) is introduced, unlike the previous nonlinearities. This phase lag is the reflection of the time delay of the backlash, which is due to the gap b. Of course, a larger b leads to a larger phase lag, which may create stability problems in feedback control systems.

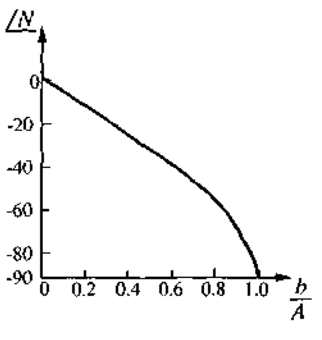


Figure 5.18 : Phase angle of describing function for backlash (degree)

5.4 Describing Function Analysis of Nonlinear Systems

For a nonlinear system containing a nonlinear element, we now know how to obtain a decribing function for the nonlinear element. The next step is to formalize the procedure in subsection 5.1.1 for the prediction of limit cycles, based on the describing function representation of the nonlinearity. The basic approach to achieve this is to apply an extended version of the famous Nyquist criterion in linear control to the equivalent system. Let us begin with a short review of the Nyquist criterion and its extension.

5.4.1 The Nyquist Criterion and Its Extension

Consider the linear system of Figure 5.19. The characteristic equation of this system is

$$\delta(p) = 1 + G(p)\, H(p) \; = 0$$

Note that $\delta(p)$, often called the *loop transfer function*, is a rational function of p, with its zeros being the poles of the closed-loop system, and its poles being the poles of the open-loop transfer function $G(p)\, H(p)$. Let us rewrite the characteristic equation as

$$G(p)\, H(p) = -1$$

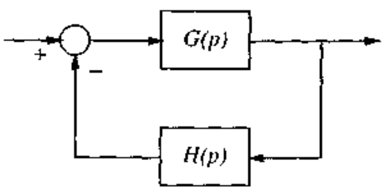


Figure 5.19 : Closed-loop linear system

Based on this equation, the famous Nyquist criterion can be derived straightforwardly from the Cauchy theorem in complex analysis. The criterion can be summarized (assuming that $G(p)\, H(p)$ has no poles or zeros on the $j\omega$ axis) in the following procedure (Figure 5.20):

1. draw, in the p plane, a so-called Nyquist path enclosing the right-half plane
2. map this path into another complex plane through $G(p)H(p)$
3. determine N, the number of clockwise encirclements of the plot of $G(p)H(p)$ around the point $(-1,0)$
4. compute Z, the number of zeros of the loop transfer function $\delta(p)$ in the right-half p plane, by

$$Z = N + P \quad , \text{ where } P \text{ is the number of unstable poles of } \delta(p)$$

Then the value of Z is the number of unstable poles of the closed-loop system.

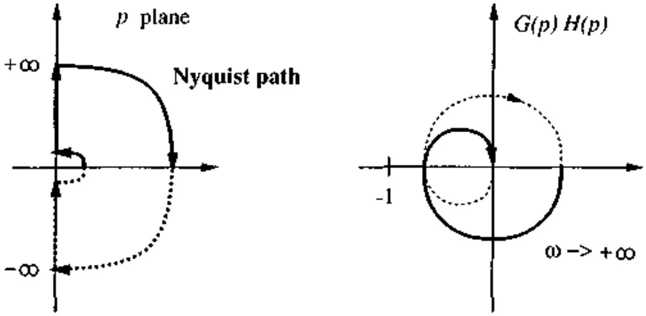


Figure 5.20 : The Nyquist criterion

A simple formal extension of the Nyquist criterion can be made to the case when a constant gain K (possibly a complex number) is included in the forward path in Figure 5.21. This modification will be useful in interpreting the stability analysis of limit cycles using the describing function method. The loop transfer function becomes

$$\delta(p) = 1 + K\, G(p)H(p)$$

with the corresponding characteristic equation

$$G(p)\, H(p) = -\, 1/K$$

The same arguments as used in the derivation of Nyquist criterion suggest the same procedure for determining unstable closed-loop poles, with the minor difference that now *Z represents the number of clockwise encirclements of the* $G(p)\ H(p)$ *plot around the point* $-\ 1/K$. Figure 5.21 shows the corresponding extended Nyquist plot.

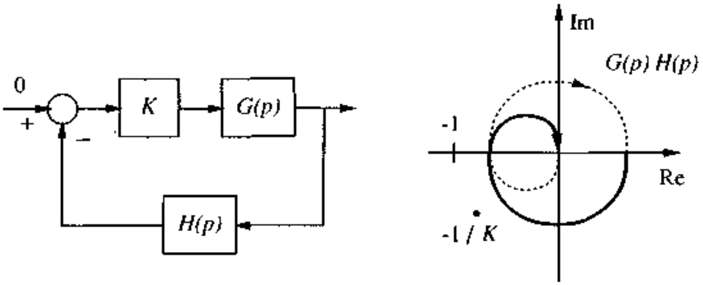


Figure 5.21 : Extension of the Nyquist criterion

5.4.2 Existence of Limit Cycles

Let us now assume that there exists a self-sustained oscillation of amplitude A and frequency ω in the system of Figure 5.22. Then the variables in the loop must satisfy the following relations:

$$x = -y$$

$$w = N(A,\omega)x$$

$$y = G(j\omega)\,w$$

Therefore, we have $y = G(j\omega)N(A,\omega)(-y)$. Because $y \neq 0$, this implies

$$G(j\omega)\,N(A,\omega) + 1 = 0 \tag{5.15}$$

which can be written as

$$G(j\omega) = -\frac{1}{N(A,\omega)} \tag{5.16}$$

Therefore, the amplitude A and frequency ω of the limit cycles in the system must satisfy (5.16). If the above equation has no solutions, then the nonlinear system has no limit cycles.

Expression (5.16) represents two nonlinear equations (the real part and imaginary part each give one equation) in the two variables A and ω. There are usually a finite number of solutions. It is generally very difficult to solve these equations by analytical methods, particularly for high-order systems, and therefore, a graphical approach is usually taken. The idea is to plot both sides of (5.16) in the complex plane and find the intersection points of the two curves.

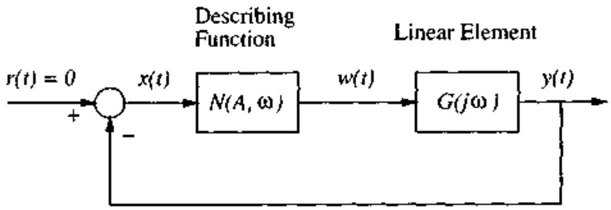


Figure 5.22 : A nonlinear system

FREQUENCY-INDEPENDENT DESCRIBING FUNCTION

First, we consider the simpler case when the describing function N being a function of the gain A only, *i.e.*, $N(A, \omega) = N(A)$. This includes all single-valued nonlinearities and some important double-valued nonlinearities such as backlash. The equality becomes

$$G(j\omega) = -\frac{1}{N(A)} \tag{5.17}$$

We can plot both the frequency response function $G(j\omega)$ (varying ω) and the negative inverse describing function $(-1/N(A))$ (varying A) in the complex plane, as in Figure 5.23. If the two curves intersect, then there exist limit cycles, and the values of A and ω corresponding to the intersection point are the solutions of Equation (5.17). If the curves intersect n times, then the system has n possible limit cycles. Which one is actually reached depends on the initial conditions. In Figure 5.23, the two curves intersect at one point K. This indicates that there is one limit cycle in the system. The amplitude of the limit cycle is A_k, the value of A corresponding to the point K on the $-1/N(A)$ curve. The frequency of the limit cycle is ω_k, the value of ω corresponding to the point K on the $G(j\omega)$ curve.

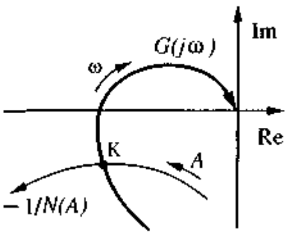


Figure 5.23 : Detection of limit cycles

Note that for single-valued nonlinearities, N is real and therefore the plot of $-1/N$ always lies on the real axis. It is also useful to point out that, as we shall discuss later, the above procedure only gives a *prediction* of the existence of limit cycles. The validity and accuracy of this prediction should be confirmed by computer simulations.

We already saw in section 5.1.1 an example of the prediction of limit cycles, for the Van der Pol equation.

FREQUENCY-DEPENDENT DESCRIBING FUNCTION

For the general case, where the describing function depends on both input amplitude and frequency ($N = N(A, \omega)$), the method can be applied, but with more complexity. Now the right-hand side of (5.15), $-1/N(A, \omega)$, corresponds to a family of curves on the complex plane with A as the running parameter and ω fixed for each curve, as shown in Figure 5.24. There are generally an *infinite* number of intersection points between the $G(j\omega)$ curve and the $-1/N(A, \omega)$ curves. Only the intersection points with matched ω indicate limit cycles.

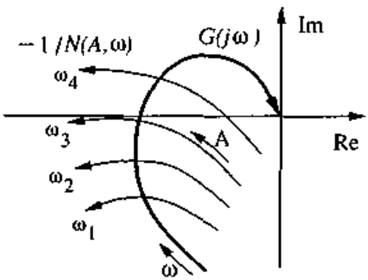


Figure 5.24 : Limit cycle detection for frequency-dependent describing functions

To avoid the complexity of matching frequencies at intersection points, it may be advantageous to consider the graphical solution of (5.16) directly, based on the plots of $G(j\omega)N(A, \omega)$. With A fixed and ω varying from 0 to ∞, we obtain a curve representing $G(j\omega)N(A,\omega)$. Different values of A correspond to a family of curves, as shown in Figure 5.25. A curve passing *through the point* $(-1,0)$ in the complex plane indicates the existence of a limit cycle, with the value of A for the curve being the amplitude of the limit cycle, and the value of ω at the point $(-1,0)$ being the frequency of the limit cycle. While this technique is much more straightforward than the previous one, it requires repetitive computation of the $G(j\omega)$ in generating the family of curves, which may be handled easily by computer.

5.4.3 Stability of Limit Cycles

As pointed out in chapter 2, limit cycles can be stable or unstable. In the above, we have discussed how to detect the existence of limit cycles. Let us now discuss how to determine the stability of a limit cycle, based on the extended Nyquist criterion in section 5.4.1.

Consider the plots of frequency response and inverse describing function in Figure 5.26. There are two intersection points in the figure, predicting that the system

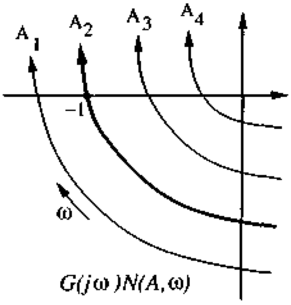


Figure 5.25 : Solving equation (5.15) graphically

has two limit cycles. Note that the value of A corresponding to point L_1 is smaller than the value of A corresponding to L_2. For simplicity of discussion, we assume that the linear transfer function $G(p)$ has no unstable poles.

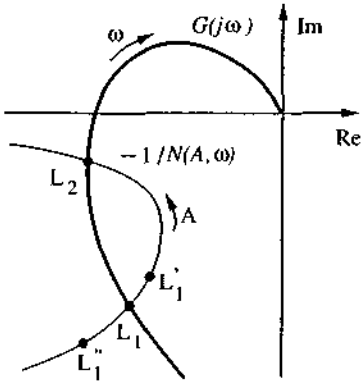


Figure 5.26 : Limit Cycle Stability

Let us first discuss the stability of the limit cycle at point L_1. Assume that the system initially operates at point L_1, with the limit cycle amplitude being A_1, and its frequency being ω_1. Due to a slight disturbance, the amplitude of the input to the nonlinear element is slightly increased, and the system operating point is moved from L_1 to L_1'. Since the new point L_1' is encircled by the curve of $G(j\omega)$, according to the extended Nyquist criterion mentioned in section 5.4.1, the system at this operating point is unstable, and the amplitudes of the system signals will increase. Therefore, the operating point will continue to move along the curve $-1/N(A)$ toward the other limit cycle point L_2. On the other hand, if the system is disturbed so that the amplitude A is decreased, with the operating point moved to the point L_1'', then A will continue to decrease and the operating point moving away from L_1 in the other direction. This is because L_1'' is not encircled by the curve $G(j\omega)$ and thus the extended Nyquist plot asserts the stability of the system. The above discussion indicates that a slight disturbance can destroy the oscillation at point L_1 and, therefore, that this limit cycle is unstable. A similar analysis for the limit cycle at point L_2 indicates that that limit cycle is stable.

Summarizing the above discussion and the result in the previous subsection, we obtain a criterion for existence and stability of limit cycles:

Limit Cycle Criterion: *Each intersection point of the curve $G(j\omega)$ and the curve $-1/N(A)$ corresponds to a limit cycle. If points near the intersection and along the increasing-A side of the curve $-1/N(A)$ are not encircled by the curve $G(j\omega)$, then the corresponding limit cycle is stable. Otherwise, the limit cycle is unstable.*

5.4.4 Reliability of Describing Function Analysis

Empirical evidence over the last three decades, and later theoretical justification, indicate that the describing function method can effectively solve a large number of practical control problems involving limit cycles. However, due to the approximate nature of the technique, it is not surprising that the analysis results are sometimes not very accurate. Three kinds of inaccuracies are possible:

1. The amplitude and frequency of the predicted limit cycle are not accurate
2. A predicted limit cycle does not actually exist
3. An existing limit cycle is not predicted

The first kind of inaccuracy is quite common. Generally, the predicted amplitude and frequency of a limit cycle always deviate somewhat from the true values. How much the predicted values differ from the true values depends on how well the nonlinear system satisfies the assumptions of the describing function method. In order to obtain accurate values of the predicted limit cycles, simulation of the nonlinear system is necessary.

The occurrence of the other two kinds of inaccuracy is less frequent but has more serious consequences. Usually, their occurrence can be detected by examining the linear element frequency response and the relative positions of the G plot and $-1/N$ plot.

Violation of filtering hypothesis: The validity of the describing function method relies on the filtering hypothesis defined by (5.5). For some linear elements, this hypothesis is not satisfied and errors may result in the describing function analysis. Indeed, a number of failed cases of describing function analysis occur in systems whose linear element has resonant peaks in its frequency response $G(j\omega)$.

Graphical Conditions: If the $G(j\omega)$ locus is tangent or almost tangent to the $-1/N$ locus, then the conclusions from a describing function analysis might be erroneous. Such an example is shown in Figure 5.27(a). This is because the effects of neglected higher harmonics or system model uncertainty may cause the change of the intersection situations, particularly when filtering in the linear element is weak. As a result, the second and third types of errors listed above may occur. A classic case of this problem involves a second-order servo with backlash studied by Nychols. While describing function analysis predicts two limit cycles (a stable one at high frequency and an unstable one at low frequency), it can be shown that the low-frequency unstable limit cycle does not exist.

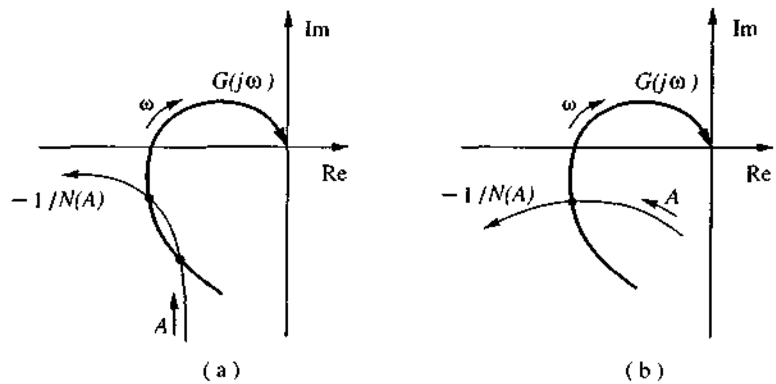


Figure 5.27 : Reliability of limit cycle prediction

Conversely, if the $-1/N$ locus intersects the G locus almost perpendicularly, then the results of the describing function are usually good. An example of this situation is shown in Figure 5.27(b).

5.5 Summary

The describing function method is an extension of the frequency response method of linear control. It can be used to *approximately* analyze and predict the behavior of important classes of nonlinear systems, including systems with hard nonlinearities. The desirable properties it inherits from the frequency response method, such as its graphical nature and the physically intuitive insights it can provide, make it an important tool for practicing engineers. Applications of the describing function method to the prediction of limit cycles were detailed. Other applications, such as predicting subharmonics, jump phenomena, and responses to external sinusoidal inputs, can be found in the literature.

5.6 Notes and References

An extensive and clear presentation of the describing function method can be found in [Gelb and VanderVelde, 1968]. A more recent treatment is contained in [Hedrick, *et al.*, 1982], which also discusses specific applications to nonlinear physical systems. The describing function method was developed and successfully used well before its mathematical justification was completely formalized [Bergen and Franks, 1971]. Figures 5.14 and 5.16 are adapted from [Shinners, 1978]. The Van der Pol oscillator example is adapted from [Hsu and Meyer, 1968].

5.7 Exercises

5.1 Determine whether the system in Figure 5.28 exhibits a self-sustained oscillation (a limit cycle). If so, determine the stability, frequency, and amplitude of the oscillation.

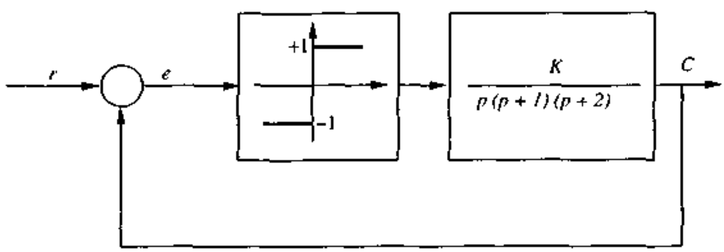


Figure 5.28 : A nonlinear system containing a relay

5.2 Determine whether the system in Figure 5.29 exhibits a self-sustained oscillation. If so, determine the stability, frequency, and amplitude of the oscillation.

5.3 Consider the nonlinear system of Figure 5.30. Determine the largest K which preserves the stability of the system. If $K = 2K_{max}$, find the amplitude and frequency of the self-sustained oscillation.

5.4 Consider the system of Figure 5.31, which is composed of a high-pass filter, a saturation function, and the inverse low-pass filter. Show that the system can be viewed as a nonlinear low-

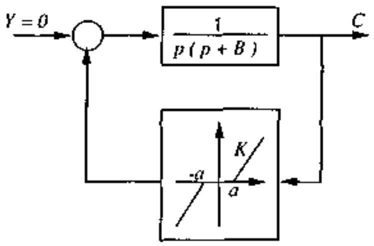


Figure 5.29 : A nonlinear system containing a dead-zone

pass filter, which attenuates high-frequency inputs *without introducing a phase lag.*

5.5 This exercise is based on a result of [Tsypkin, 1956].

Consider a nonlinear system whose output $w(t)$ is related to the input $u(t)$ by an odd function, of the form

$$w(t) = F(u(t)) = -F(-u(t)) \tag{5.18}$$

Derive the following very simple approximate formula for the describing function $N(A)$

$$N(A) \approx \frac{2}{3A}\,[\,F(A) + F(A/2)\,]$$

To this effect, you may want to use the fact that

$$\frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{1}{6}\,[\,f(1) + f(-1) + 2f(1/2) + 2f(-1/2)\,] + R$$

where the remainder R verifies $R = f^6(\xi)/(2^5\,6!)$ for some $\xi \in (-1, 1)$. Show that approximation (5.18) is quite precise (how precise?).

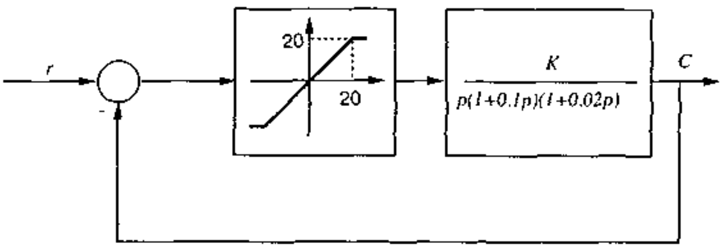


Figure 5.30 : A nonlinear system containing a saturation

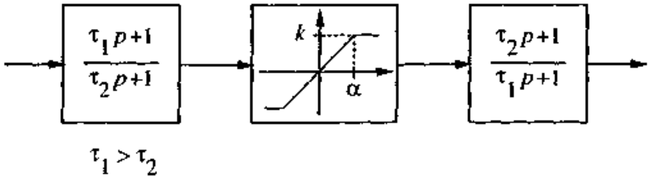


Figure 5.31 : A nonlinear low-pass filter

Invert (5.18) so as to obtain for the input-output relation a solution of the form

$$F(A) \approx \sum_{k=0}^{\infty} (-1)^k \, \frac{3\,A}{2^{k+1}} \, N(\frac{A}{2^k})$$

5.6 In this exercise, adapted form [Phillips and Harbor, 1988], let us consider the system of Figure 5.32, which is typical of the dynamics of electronic oscillators used in laboratories, with

$$G(p) = \frac{-5\,p}{p^2 + p + 25}$$

Use describing function analysis to predict whether the system exhibits a limit cycle, depending on the value of the saturation level k. In such cases, determine the limit cycle's frequency and amplitude.

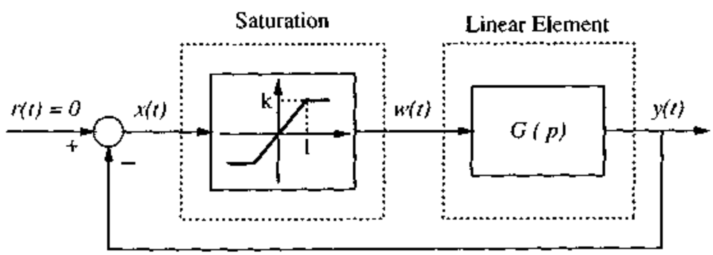


Figure 5.32 : Dynamics of an electronic oscillator

Interpret intuitively, by assuming that the system is started at some small initial state, and noticing that $y(t)$ can stay neither at small values (because of instability) nor at saturation values (by applying the final value theorem of linear control).

Part II
Nonlinear Control Systems Design

In Part I, we studied how to analyze the behavior of a nonlinear control system, assuming that the control system had been designed. Part II is devoted to the problem of designing nonlinear control systems. In this introduction, we discuss some general issues involved in nonlinear control system design, particularly emphasizing the differences of nonlinear control design problems from linear ones. In the following chapters, we will detail the specific control methods available to the designer.

As pointed out in chapter 1, the objective of control design can be stated as follows: *given a physical system to be controlled and the specifications of its desired behavior, construct a feedback control law to make the closed-loop system display the desired behavior.* In accordance with this design objective, we consider a number of key issues. First, two basic types of nonlinear control problems, nonlinear regulation and nonlinear tracking, are defined. Next, the specifications of the desired behavior of nonlinear control systems are discussed. Basic issues in constructing nonlinear controllers are then outlined. Finally, the major methods available for designing nonlinear controllers are briefly surveyed.

II.1 Nonlinear Control Problems

If the tasks of a control system involve large range and/or high speed motions, nonlinear effects will be significant in the dynamics and nonlinear control may be necessary to achieve the desired performance. Generally, the tasks of control systems can be divided into two categories: stabilization (or regulation) and tracking (or servo). In stabilization problems, a control system, called a *stabilizer* (or a regulator), is to be designed so that the state of the closed-loop system will be stabilized around an equilibrium point. Examples of stabilization tasks are temperature control of refrigerators, altitude control of aircraft and position control of robot arms. In tracking control problems, the design objective is to construct a controller, called a *tracker*, so that the system output tracks a given time-varying trajectory. Problems such as making an aircraft fly along a specified path or making a robot hand draw straight lines or circles are typical tracking control tasks.

STABILIZATION PROBLEMS

In order to facilitate the analytic study of stabilization and tracking design in the later chapters, let us provide some formal definitions of stabilization and tracking problems.

Asymptotic Stabilization Problem: *Given a nonlinear dynamic system described by*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

find a control law $\mathbf{u}$ *such that, starting from anywhere in a region in* Ω, *the state* $\mathbf{x}$ *tends to* $\mathbf{0}$ *as* $t \rightarrow \infty$.

If the control law depends on the measurement signals directly, it is said to be a static control law. If it depends on the measurement through a differential equation, the control law is said to be a dynamic control law, *i.e.*, there is dynamics in the control law. For example, in linear control, a proportional controller is a static controller, while a lead-lag controller is a dynamic controller.

Note that, in the above definition, we allow the size of the region Ω to be large; otherwise, the stabilization problem may be adequately solved using linear control. Note also that if the objective of the control task is to drive the state to some non-zero set-point $\mathbf{x}_d$, we can simply transform the problem into a zero-point regulation problem by taking $\mathbf{x} - \mathbf{x}_d$ as the state.

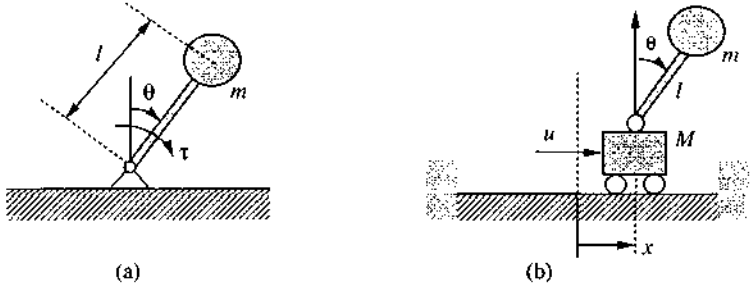


Figure II.1 : (a) a pendulum; (b) an inverted pendulum, with cart

Example II.1: Stabilization of a pendulum

Consider the pendulum in Figure II.1(a). Its dynamics is

$$J\ddot{\theta} - mgl\sin\theta = \tau \tag{II.1}$$

Assume that our task is to bring the pendulum from a large initial angle, say $\theta(0) = 60^o$, to the *vertical-up* position. One choice of the stabilizer is

$$\tau = -k_d\dot{\theta} - k_p\theta - mgl\sin\theta \tag{II.2}$$

with k_d and k_p denoting positive constants. This leads to the following globally stable closed-loop dynamics

$$J\ddot{\theta} + k_d\dot{\theta} + k_p\theta = 0$$

i.e., the controlled pendulum behaves as a stable mass-spring-damper system. Note that the controller (II.2) is composed of a P.D. (proportional plus derivative) feedback part for stability and a feedforward part for gravity compensation. Another interesting controller is

$$\tau = -k_d\dot{\theta} - 2mgl\sin\theta \tag{II.3}$$

which leads to the stable closed-loop dynamics

$$J\ddot{\theta} + k_d\dot{\theta} + mgl\sin\theta = 0$$

This amounts to artificially reverting the gravity field and adding viscous damping.

This example illustrates the point that feedback and feedforward control actions amount to modifying the dynamics of the plant into a desirable form. □

However, many nonlinear stabilization problems are not so easy to solve. One such example is the inverted pendulum shown in Figure II.1(b) which can be easily shown to have the following dynamics

$$(M+m)\ddot{x} + ml\cos\theta\,\ddot{\theta} - ml\sin\theta\,\dot{\theta}^2 = u \qquad \text{(II.4a)}$$

$$m\ddot{x}\cos\theta + ml\,\ddot{\theta} - mg\sin\theta = 0 \qquad \text{(II.4b)}$$

(where the mass of the cart is not assumed to be negligible). A particularly interesting task is to design a controller to bring the inverted pendulum from a vertical-down position at the middle of the lateral track to a vertical-up position at the same lateral point. This seeming simple nonlinear control problem is surprisingly difficult to solve in a systematic fashion (see Exercise II.5). This problem arises because there are two degrees of freedom and only one input.

TRACKING PROBLEMS

The task of asymptotic tracking can be defined similarly.

Asymptotic Tracking Problem: *Given a nonlinear dynamics system described by*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x})$$

and a desired output trajectory $\mathbf{y}_d$, *find a control law for the input* $\mathbf{u}$ *such that, starting from any initial state in a region* $\mathbf{\Omega}$, *the tracking errors* $\mathbf{y}(t) - \mathbf{y}_d(t)$ *go to zero, while the whole state* $\mathbf{x}$ *remains bounded.*

Note that, from a practical point of view, one may require that $\mathbf{x}$ actually remain "reasonably" bounded, and, in particular, within the range of validity of the system model. This may be verified either analytically, or in simulations.

When the closed-loop system is such that proper initial states imply zero tracking error for all the time,

$$\mathbf{y}(t) \equiv \mathbf{y}_d(t) \qquad \forall\, t \geq 0$$

the control system is said to be capable of *perfect tracking*. Asymptotic tracking implies that perfect tracking is asymptotically achieved. Exponential tracking convergence can be defined similarly.

Throughout the rest of the book, unless otherwise specified, we shall make the mild assumption that the desired trajectory y_d and its derivatives up to a sufficiently high order (generally equal to the system's order) are continuous and bounded. We also assume that $y_d(t)$ *and its derivatives* are available for on-line control computation. This latter assumption is satisfied by control tasks where the desired output $y_d(t)$ is planned ahead of time. For example, in robot tracking tasks, the desired position history is generally planned ahead of time and its derivatives can be easily obtained.

Actually, smooth time-histories are often generated themselves through a filtering process, thereby automatically providing higher derivatives of the desired output. In some tracking tasks, however, the assumption is not satisfied, and a so-called reference model may be used to provide the required derivative signals. For example, in designing a tracking control system for the antenna of a radar so that it will closely point toward an aircraft at all times, we only have the position of the aircraft $y_a(t)$ available at a given time instant (assuming that it is too noisy to be numerically differentiated). However, generally the tracking control law will also use the derivatives of the signals to be tracked. To solve this problem, we can generate the desired position, velocity and acceleration to be tracked by the antenna using the following second-order dynamics

$$\ddot{y}_d + k_1 \dot{y}_d + k_2 y_d = k_2 y_a(t) \qquad \text{(II.5)}$$

where k_1 and k_2 are chosen positive constants. Thus the problem of following the aircraft is translated into the problem of tracking the output $y_d(t)$ of the reference model. Note that the reference model serves the dual purpose of providing the desired output of the tracking system in response to the aircraft position measurements, and generating the derivatives of the desired output for tracker design. Of course, for the approach to be effective, the filtering process described by (II.5) should be fast enough for $y_d(t)$ to closely approximate $y_a(t)$.

For non-minimum phase systems (precise definitions of nonlinear non-minimum phase systems will be provided in chapter 6), perfect tracking and asymptotic tracking cannot be achieved, as seen in the following example.

Example II.2: Tracking control of a non-minimum phase linear system

Consider the linear system

$$\ddot{y} + 2\dot{y} + 2y = -\dot{u} + u$$

The system is non-minimum phase because it has a zero at $p = 1$. Assume that perfect tracking is achieved, *i.e.*, that $y(t) \equiv y_d(t)$, $\forall\, t \geq 0$. Then, the input u satisfies

$$\dot{u} - u = -(\ddot{y}_d + 2\dot{y}_d + 2y_d)$$

Since this represents an unstable dynamics, u diverges exponentially. Note that the above dynamics has a pole which exactly coincides with the unstable zero of the original system, *i.e.*, perfect tracking for non-minimum phase systems can be achieved only by infinite control inputs. By writing u as

$$u = -\frac{p^2 + 2p + 2}{p - 1}\, y_d$$

we see that the perfect-tracking controller is actually inverting the plant dynamics. □

The inability of perfect tracking for a non-minimum phase linear system has its roots in its inherent tendency of "undershooting" in its step response. Thus, the control design objective for non-minimum phase systems should not be perfect tracking or asymptotic tracking. Instead, we should be satisfied with, for example, bounded-error tracking, with small tracking error being achieved for desired trajectories of particular interest.

RELATIONS BETWEEN STABILIZATION AND TRACKING PROBLEMS

Normally, tracking problems are more difficult to solve than stabilization problems, because in tracking problems the controller should not only keep the whole state stabilized but also drive the system output toward the desired output. However, from a theoretical point of view, tracking design and stabilization design are often related. For instance, if we are to design a tracker for the plant

$$\ddot{y} + f(\dot{y}, y, u) = 0$$

so that $e(t) = y(t) - y_d(t)$ goes to zero, the problem is equivalent to the asymptotic stabilization of the system

$$\ddot{e} + f(\dot{e}, e, u, y_d, \dot{y}_d, \ddot{y}_d) = 0 \tag{II.6}$$

whose state components are e and $\dot{e}$. Clearly, the tracker design problem is solved if we know how to design a stabilizer for the non-autonomous dynamics (II.6).

On the other hand, stabilization problems can often be regarded as a special case of tracking problems, with the desired trajectory being a constant. In model reference control, for instance, a set-point regulation problem is transformed into a tracking problem by incorporating a reference model to filter the supplied set-point value and generate a time-varying output as the ideal response for the tracking control system.

II.2 Specifying the Desired Behavior

In linear control, the desired behavior of a control system can be *systematically* specified, either in the time-domain (in terms of rise time, overshoot and settling time corresponding to a step command) or in the frequency domain (in terms of regions in which the loop transfer function must lie at low frequencies and at high frequencies). In linear control design, one first lays down the quantitative specifications of the closed-loop control system, and then synthesizes a controller which meets these specifications. However, systematic specification for nonlinear systems (except those equivalent to linear systems) is much less obvious because the response of a nonlinear system to one command does not reflect its response to another command, and furthermore a frequency-domain description is not possible.

As a result, for nonlinear systems, one often looks instead for some *qualitative* specifications of the desired behavior in the operating region of interest. Computer simulation is an important complement to analytical tools in determining whether such specifications are met. Regarding the desired behavior of nonlinear control systems, a designer can consider the following characteristics:

Stability must be guaranteed for the nominal model (the model used for design), either in a local sense or in a global sense. The region of stability and convergence are also of interest.

Accuracy and speed of response may be considered for some "typical" motion trajectories in the region of operation. For some classes of systems, appropriate controller design can actually guarantee consistent tracking accuracy independently of the desired trajectory, as discussed in chapter 7.

Robustness is the sensitivity to effects which are not considered in the design, such as disturbances, measurement noise, unmodeled dynamics, *etc*. The system should be able to withstand these neglected effects when performing the tasks of interest.

Cost of a control system is determined mainly by the number and type of actuators, sensors, and computers necessary to implement it. The actuators, sensors and the controller complexity (affecting computing requirement) should be chosen consistently and suit the particular application.

A couple of remarks can be made at this point. First, stability does not imply the ability to withstand persistent disturbances of even small magnitude (as discussed in section 4.9.2). The reason is that stability of a nonlinear system is defined with respect to initial conditions, and only temporary disturbances may be translated as initial conditions. For example, a stable control system may guarantee an aircraft's ability to withstand gusts, while being inept at handling windshears even of small magnitude. This situation is different from that of linear control, where stability always implies ability to withstand bounded disturbances (assuming that the system does stay in its linear range). The effects of persistent disturbance on nonlinear system behavior are addressed by the concept of robustness. Secondly, the above qualities conflict to some extent, and a good control system can be obtained only based on effective trade-offs in terms of stability/robustness, stability/performance, cost/performance, and so on.

II.3 Some Issues In Constructing Nonlinear Controllers

We now briefly describe a few aspects of controller design.

A Procedure for Control Design

Given a physical system to be controlled, one typically goes through the following standard procedure, possibly with a few iterations:

1. specify the desired behavior, and select actuators and sensors;
2. model the physical plant by a set of differential equations;
3. design a control law for the system;
4. analyze and simulate the resulting control system;
5. implement the control system in hardware.

Experience, creativity, and engineering judgment are all important in this process. One should consider integrating control design with plant design if possible, similarly to the many advances in aircraft design which have been achieved using linear control techniques. Sometimes the addition or relocation of actuators and sensors may make an otherwise intractable nonlinear control problem easy.

Modeling Nonlinear Systems

Modeling is basically the process of constructing a mathematical description (usually a set of differential equations) for the physical system to be controlled. Two points can be made about modeling. First, one should use good understanding of the system dynamics and the control tasks to obtain a tractable yet accurate model for control design. Note that more accurate models are not always better, because they may require unnecessarily complex control design and analysis and more demanding computation. The key here is to keep "essential" effects and discard insignificant effects in the system dynamics in the operating range of interest. Second, modeling is more than obtaining a nominal model for the physical system: it should also provide some characterization of the model uncertainties, which may be used for robust design, adaptive design, or merely simulation.

Model uncertainties are the differences between the model and the real physical system. Uncertainties in parameters are called parametric uncertainties while the others are called non-parametric uncertainties. For example, for the model of a controlled mass

$$m\ddot{x} = u$$

the uncertainty in m is parametric uncertainty, while the neglected motor dynamics, measurement noise, sensor dynamics are non-parametric uncertainties. Parametric uncertainties are often easy to characterize. For example, m may be known to lie somewhere between 2 kg and 5 kg. The characterization of unmodeled dynamics for nonlinear systems is often more difficult, unlike the linear control case where frequency-domain characterizations can be systematically applied.

Feedback and Feedforward

In nonlinear control, the concept of feedback plays a fundamental role in controller design, as it does in linear control. However, the importance of feedforward is much more conspicuous than in linear control. Feedforward is used to cancel the effects of known disturbances and provide anticipative actions in tracking tasks. Very often it is impossible to control a nonlinear system stably without incorporating feedforward action in the control law. Note that a model of the plant is always required for feedforward compensation (although the model need not be very accurate).

Asymptotic tracking control always requires feedforward actions to provide the forces necessary to make the required motion. It is interesting to note that many tracking controllers can be written in the form

$$\mathbf{u} = \text{feedforward} + \text{feedback}$$

or in a similar form. The feedforward part intends to provide the necessary input for following the specified motion trajectory and canceling the effects of the known disturbances. The feedback part then stabilizes the tracking error dynamics.

As an illustration of the use of feedforward, let us consider tracking controller design in the familiar context of linear systems (as applicable to devices such as an x-y plotter, for instance). The discussion is interesting in its own right, since tracking of time-varying trajectories is not commonly emphasized in linear control texts.

Example II.3: Tracking control of linear systems

Consider a linear (controllable and observable) minimum-phase system in the form

$$A(p)\,y = B(p)\,u \qquad \text{(II.7)}$$

where

$$A(p) = a_o + a_1 p + ... + a_{n-1} p^{n-1} + p^n$$

$$B(p) = b_o + b_1 p + ... + b_m p^m$$

The control objective is to make the output $y(t)$ follow a time-varying desired trajectory $y_d(t)$. We assume that only the output $y(t)$ is measured, and that $y_d, \dot{y}_d, ..., y_d^{(r)}$ are known, with r being the relative degree (the excess of poles over zeros) of the transfer function (thus, $r = n - m$).

The control design can be achieved in two steps. First, let us take the control law in the form of

$$u = v + \frac{A(p)}{B(p)} y_d \tag{II.8}$$

where v is a new input to be determined. Substitution of (II.8) into (II.7) leads to

$$A(p)\,e = B(p)\,v \tag{II.9}$$

where $e(t) = y(t) - y_d(t)$ is the tracking error. The feedforward signal $(A/B)\, y_d$ can be computed as

$$\frac{A}{B} y_d = \alpha_1 y_d^{(r)} + ... + \alpha_r y_d + w$$

where the α_i ($i = 1, ... , r$) are constants obtained from dividing A by B, and $w(t)$ is a filtered version of $y_d(t)$.

The second step is to construct input u so that the error dynamics is asymptotically stable. Since e is known (by subtracting the known y_d from the measured y), while its derivatives are not, one can stabilize e by using standard linear techniques, *e.g.*, pole-placement together with a *Luenberger observer. A simpler way of deriving the control law is to let*

$$v = \frac{C(p)}{D(p)} e \tag{II.10}$$

with C and D being polynomials of order $(n - m)$. With this control law, the closed loop dynamics is

$$(AC + BD)\,e = 0$$

If the coefficients of C and D are chosen properly, the poles of the closed-loop polynomial can be placed anywhere in the complex plane (with the complex poles in conjugate pairs), as we shall see in chapter 8. Therefore, the control law

$$u \ = \ \frac{A}{B} y_d \ + \ \frac{C}{D} e \tag{II.11}$$

will guarantee that the tracking error $e(t)$ remains at zero if initial conditions satisfy $y^{(i)}(0) = y_d^{(i)}(0)$ $(i = 1, ... , r)$, and exponentially converges to zero if the initial conditions do not satisfy these conditions.

The block diagram of the closed-loop system is depicted in Figure II.3. We can make the following comments about the control system:

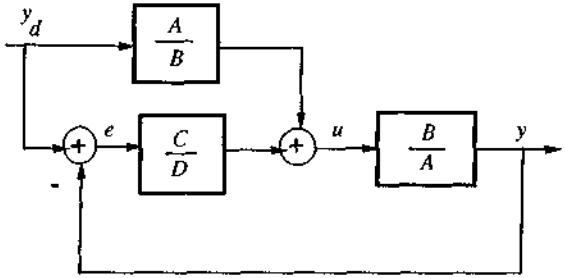


Figure II.2 : Linear Tracking Control system

• The feedforward part of the control law, computed by inverting the plant model, is responsible for reducing and eliminating the tracking errors, while the feedback part results in stability of the whole system. If some derivatives of the desired trajectories $y_d(t)$ are not available, one can simply omit them from the feedforward, which will only cause bounded error in tracking. Note that one may easily adapt the above controller for model reference control, with y_d and its derivatives provided by the reference model.

• The control law (II.10) is equivalent to implementing a reduced-order Luenberger observer. Higher order observers can also be used, possibly for the purpose of increasing system robustness by exploiting the added flexibility.

• The above method cannot be directly used for tracking control of non-minimum phase systems (with some of the roots of $B(p)$ having positive real parts) since the inverse model A/B is unstable. However, by feedforwarding low-frequency components of the desired trajectories, good tracking in the low-frequency range (lower than the left-half plane zeros of the plant) may still be achieved. For instance, by using $(A/B_1)\, y_d$ as the feedforward signal, the tracking error can be easily found to be

$$e(t) = \frac{C}{AC + BD}\left[\frac{B}{B_1} - 1\right] A\, y_d$$

If B_1 is close to B at low frequencies, the control system can track a slowly varying $y_d(t)$ well. One particular choice of B_1 is to eliminate the right half-plane zeros of B, which will lead to good tracking for desired trajectories with frequencies lower than the right half-plane zeroes. □

Importance of Physical Properties

In linear control, it is common practice to generate a set of differential equations for a physical system, and then forget where they came from. This presents no major problem there, at least in a theoretical sense, because linear control theory provides powerful tools for analysis and design. However, such a procedure is typically undesirable for nonlinear systems, because the number of tools available for attacking nonlinear problems is comparatively limited. In nonlinear control design, *exploitation of the physical properties* can sometimes make control design for complex nonlinear plants a simple issue, or may easily solve an otherwise intractable design problem. This point is forcibly demonstrated in the solution of the adaptive robot control problem. Adaptive control of robot manipulators was long recognized to be far out of the reach of conventional adaptive control theory, because a robot's dynamics is strongly nonlinear and has multiple inputs. However, the use of two physical facts, namely, the positive definiteness of the inertia matrix and the possibility of linearly parametrizing robot dynamics, successfully led to an adaptive controller with the desirable properties of global stability and tracking convergence, as shown in chapter 9.

Discrete Implementations

As discussed in chapter 1, nonlinear physical systems are continuous in nature and are hard to meaningfully discretize, while digital control systems may be treated as continuous-time systems in analysis and design if high sampling rates are used (specific quantifications are discussed, *e.g.*, in section 7.3). Thus, we perform nonlinear system analysis and controller design in continuous-time form. However, of course, the control law is generally implemented digitally.

Numerical integration and differentiation are sometimes explicit parts of a controller design. Numerical differentiation may avoid the complexity of constructing the whole system state based on partial measurements (the nonlinear observer problem), while numerical integration is a standard component of most adaptive controller designs, and can also be needed more generally in dynamic controllers.

Numerical differentiation may be performed in many ways, all aimed at getting a reasonable estimate of the *time-derivative*, while at the same time avoiding the generation of large amounts of noise. One can use, for instance, a filtered differentiation of the form

$$\frac{\alpha p}{p+\alpha} = \alpha\,\frac{p+\alpha-\alpha}{p+\alpha} = \alpha\left(1 - \frac{\alpha}{p+\alpha}\right)$$

where p is the Laplace variable and $\alpha >> 1$. The discrete implementation of the above equation, assuming *e.g.*, a zero-order hold, is simply

$$y_{new} = a_1\, y_{old} + a_2\, x \qquad\qquad \dot{x}_{new} = \alpha\,(x - y_{new}) \tag{II.12}$$

where the constants a_1 and a_2 are defined as

$$a_1 = e^{-\alpha T} \qquad a_2 = 1 - a_1$$

and T is the sampling period. Note that this approximate procedure can actually be interpreted as building a reduced-order observer for the system. However, it does not use an explicit model, so that the system can be nonlinear and its parameters unknown. An alternative choice of filter structure is studied in Exercise II.6.

Numerical integration actually consists in *simulating in real-time* some (generally nonlinear) dynamic subcomponent required by the controller. Many methods can again be used. The simplest, which can work well at high sampling rates or for low-dimensional systems, is the so-called *Euler integration*

$$x_{new} = x_{old} + \dot{x}\, T$$

where T is the sampling period. A more sophisticated approach, which is very effective in most cases but is more computationally involved than mere Euler integration, is two-step *Adams-Bashforth* integration

$$x(t) = x(t-T) + \left(\frac{3}{2}\dot{x}(t-T) - \frac{1}{2}\dot{x}(t-2T)\right) T$$

More complex techniques may also be used, depending on the desired trade-off between accuracy and on-line computational efficiency.

II.4 Available Methods of Nonlinear Control Design

As in the analysis of nonlinear control systems, there is no general method for designing nonlinear controllers. What we have is a rich collection of alternative and complementary techniques, each best applicable to particular classes of nonlinear control problems.

Trial-and-error

Based on the analysis methods provided in Part I, one can use trial-and-error to synthesize controllers, similarly to, *e.g.*, linear lead-lag controller design based on Bode plots. The idea is to use the analysis tools to guide the search for a controller which can then be justified by analysis and simulations. The phase plane method, the describing function method, and Lyapunov analysis can all be used for this purpose. Experience and intuition are critical in this process. However, for complex systems trial-and-error often fails.

Feedback linearization

As discussed earlier, the first step in designing a control system for a given physical plant is to derive a meaningful *model* of the plant, *i.e.*, a model that captures the key dynamics of the plant in the operational range of interest. Models of physical systems come in various forms, depending on the modeling approach and assumptions. Some forms, however, lend themselves more easily to controller design. Feedback linearization deals with techniques for *transforming original system models into equivalent models of a simpler form.*

Feedback linearization can be used as a nonlinear design methodology. The basic idea is to first transform a nonlinear system into a (fully or partially) linear system, and then use the well-known and powerful linear design techniques to complete the control design. The approach has been used to solve a number of practical nonlinear control problems. It applies to important classes of nonlinear systems (so-called input-state linearizable or minimum-phase systems), and typically requires full state measurement. However, it does not guarantee robustness in the face of parameter uncertainty or disturbances.

Feedback linearization techniques can also be used as model-simplifying devices for robust or adaptive controllers, to be discussed next.

Robust control

In pure model-based nonlinear control (such as the basic feedback linearization control approach), the control law is designed based on a nominal model of the physical system. How the control system will behave in the presence of model uncertainties is not clear at the design stage. In robust nonlinear control (such as, *e.g.*, sliding control), on the other hand, the controller is designed based on the consideration of both the the nominal model *and* some characterization of the model uncertainties (such as the knowledge that the load to be picked up and carried by a robot is between 2 kg and 10 kg). Robust nonlinear control techniques have proven very effective in a variety of practical control problems. They apply best to specific classes of nonlinear systems, and generally require state measurements.

Adaptive control

Adaptive control is an approach to dealing with uncertain systems or time-varying systems. Although the term "adaptive" can have broad meanings, current adaptive control designs apply mainly to systems with known dynamic structure, but unknown constant or slowly-varying parameters. Adaptive controllers, whether developed for linear systems or for nonlinear systems, are inherently nonlinear.

Systematic theories exist for the adaptive control of linear systems. Existing adaptive control techniques can also treat important classes of nonlinear systems, with measurable states and linearly parametrizable dynamics. For these nonlinear systems, adaptive control can be viewed as an alternative and complementary approach to robust nonlinear control techniques, with which it can be combined effectively. Although most adaptive control results are for single-input single-output systems, some important nonlinear physical systems with multiple-inputs have also been studied successfully.

Gain-scheduling

Gain scheduling (see [Rugh, 1991] for a recent discussion, and references therein) is an attempt to apply the well developed linear control methodology to the control of nonlinear systems. It was originally developed for the trajectory control of aircraft. The idea of gain-scheduling is to select a number of operating points which cover the range of the system operation. Then, at each of these points, the designer makes a linear time-invariant approximation to the plant dynamics and designs a linear controller for each linearized plant. Between operating points, the parameters of the compensators are then interpolated, or *scheduled*, thus resulting in a global compensator. Gain scheduling is conceptually simple, and, indeed, practically successful for a number of applications. The main problem with gain scheduling is that has only limited theoretical guarantees of stability in nonlinear operation, but uses some loose practical guidelines such as "the scheduling variables should change slowly" and "the scheduling variables should capture the plant's nonlinearities". Another problem is the computational burden involved in a gain-scheduling design, due to the necessity of computing many linear controllers.

II.5 Exercises

II.1 Why do linear systems with a right half-plane zero exhibit the so-called "undershooting" phenomenon (the step response initially goes downward)? Is the inability of perfect tracking for non-minimum phase systems related to the undershooting phenomenon?

Consider for instance the system

$$y(p) = \frac{(1-p)}{p^2+2p+2}\,u$$

Sketch its step response and compare it with that of the system

$$y(p) = \frac{(1+p)}{p^2+2p+2}\,u$$

What are the differences in frequency responses?

What does the step response of a non-minimum phase linear system look like if it has two right half-plane zeros? Interpret and comment.

II.2 Assume that you are given a pendulum and the task of designing a control system to track the desired trajectory

$$\theta_d(t) = A \sin\omega t \qquad 0 < A \leq 90^o \qquad 0 < \omega \leq 10\, Hz$$

What hardware components do you need to implement the control system? What requirements does the task impose on the the specifications of the components? Provide a detailed outline of your control system design.

II.3 List the model uncertainties associated with the pendulum model (II.1). Discuss how to characterize them.

II.4 Carry out the tracking design for the linear plants

$$y(p) = \frac{(3+p)}{p^2+2p+2}\,u \qquad\qquad y(p) = \frac{(3-p)}{p^2+2p+2}\,u$$

Simulate their responses to the desired trajectories

$$y_d(t) = \sin\omega t$$

with ω being 0.5, 1.5, and 4. rad/sec.

II.5 Figure out an energy-based strategy to bring the inverted pendulum in Figure II.1.b from the vertical-down position to the vertical-up position. (*Hint*: You may want first to express the system's kinetic energy in a modified coordinate system chosen such that rotation and translation are uncoupled.)

What does your controller guarantee along the x direction? Does it reduce to the usual linear inverted pendulum controller when linearized?

II.6 An alternative to the filtered differentiation (II.12) consists in simply passing an approximate derivative through a zero-order hold discrete filter, *e.g.*,

$$\dot{x}_{new} = c_1\,\dot{x}_{old} + (1-c_1)\,\frac{x - x_{old}}{T}$$

where $c_1 = e^{-\alpha T}$. Discuss the relative merits of the two approaches.

Chapter 6
Feedback Linearization

Feedback linearization is an approach to nonlinear control design which has attracted a great deal of research interest in recent years. The central idea of the approach is to algebraically transform a nonlinear system dynamics into a (fully or partly) linear one, so that linear control techniques can be applied. This differs entirely from conventional linearization (*i.e.*, Jacobian linearization, as in section 3.3) in that feedback linearization is achieved by exact state transformations and feedback, rather than by linear approximations of the dynamics.

The idea of simplifying the form of a system's dynamics by choosing a different state representation is not entirely unfamiliar. In mechanics, for instance, it is well known that the form and complexity of a system model depend considerably on the choice of reference frames or coordinate systems. Feedback linearization techniques can be viewed as ways of *transforming original system models into equivalent models of a simpler form*. Thus, they can also be used in the development of robust or adaptive nonlinear controllers, as discussed in chapters 7 and 8.

Feedback linearization has been used successfully to address some practical control problems. These include the control of helicopters, high performance aircraft, industrial robots, and biomedical devices. More applications of the methodology are being developed in industry. However, there are also a number of important shortcomings and limitations associated with the feedback linearization approach. Such problems are still very much topics of current research.

This chapter provides a description of feedback linearization, including what it is, how to use it for control design and what its limitations are. In section 6.1, the basic concepts of feedback linearization are described intuitively and illustrated with simple examples. Section 6.2 introduces mathematical tools from differential geometry which are useful to generalize these concepts to a broad class of nonlinear systems. Sections 6.3 and 6.4 describe feedback linearization theory for SISO systems, and section 6.5 extends the methodology to MIMO systems.

6.1 Intuitive Concepts

This section describes the basic concepts of feedback linearization intuitively, using simple examples. The following sections will formalize these concepts for more general nonlinear systems.

6.1.1 Feedback Linearization And The Canonical Form

In its simplest form, feedback linearization amounts to canceling the nonlinearities in a nonlinear system so that the closed-loop dynamics is in a linear form. This very simple idea is demonstrated in the following example.

Example 6.1: Controlling the fluid level in a tank

Consider the control of the level h of fluid in a tank (Figure 6.1) to a specified level h_d. The control input is the flow u into the tank, and the initial level is h_o.

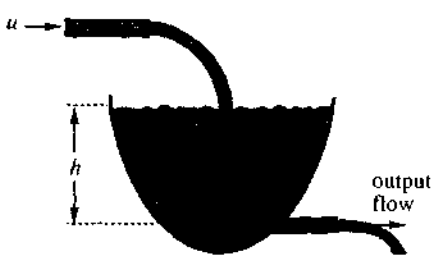


Figure 6.1 : Fluid level control in a tank

The dynamic model of the tank is

$$\frac{d}{dt}\left[\int_0^h A(h)\,dh\right] = u(t) - a\sqrt{2gh} \tag{6.1}$$

where $A(h)$ is the cross section of the tank and a is the cross section of the outlet pipe. If the initial level h_o is quite different from the desired level h_d, the control of h involves a nonlinear regulation problem.

The dynamics (6.1) can be rewritten as

$$A(h)\,\dot{h} = u - a\sqrt{2gh}$$

If $u(t)$ is chosen as

$$u(t) = a\sqrt{2gh} + A(h)\,v \tag{6.2}$$

with v being an "equivalent input" to be specified, the resulting dynamics is linear

$$\dot{h} = v$$

Choosing v as

$$v = -\alpha\tilde{h} \tag{6.3}$$

with $\tilde{h} = h(t) - h_d$ being the level error, and α being a strictly positive constant, the resulting closed loop dynamics is

$$\dot{h} + \alpha\tilde{h} = 0 \tag{6.4}$$

This implies that $\tilde{h}(t) \rightarrow 0$ as $t \rightarrow \infty$. Based on (6.2) and (6.3), the actual input flow is determined by the nonlinear control law

$$u(t) = a\sqrt{2gh} - A(h)\,\alpha\tilde{h} \tag{6.5}$$

Note that, in the control law (6.5), the first part on the right-hand side is used to provide the output flow $a\sqrt{2gh}$, while the second part is used to raise the fluid level according to the the desired linear dynamics (6.4).

Similarly, if the desired level is a known time-varying function $h_d(t)$, the equivalent input v can be chosen as

$$v = \dot{h}_d(t) - \alpha\tilde{h}$$

so as to still yield $\tilde{h}(t) \rightarrow 0$ as $t \rightarrow \infty$. □

The idea of feedback linearization, *i.e.*, of canceling the nonlinearities and imposing a desired linear dynamics, can be simply applied to a class of nonlinear systems described by the so-called *companion form*, or *controllability canonical form*. A system is said to be in companion form if its dynamics is represented by

$$x^{(n)} = f(\mathbf{x}) + b(\mathbf{x})\,u \tag{6.6}$$

where u is the scalar control input, x is the scalar output of interest, $\mathbf{x} = [\, x\, , \dot{x}\, , \; \ldots \; , x^{(n-1)}\,]^T$ is the state vector, and $f(\mathbf{x})$ and $b(\mathbf{x})$ are nonlinear functions of the states. This form is unique in the fact that, although derivatives of x appear in this equation, no derivative of the input u is present. Note that, in state-space representation, (6.6) can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ \cdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ \cdots \\ x_n \\ f(\mathbf{x}) + b(\mathbf{x})\, u \end{bmatrix}$$

For systems which can be expressed in the controllability canonical form, using the control input (assuming b to be non-zero)

$$u = \frac{1}{b}\, [\, v - f\,] \tag{6.7}$$

we can cancel the nonlinearities and obtain the simple input-output relation (multiple-integrator form)

$$x^{(n)} = v$$

Thus, the control law

$$v = -\, k_o\, x - k_1 \dot{x} - \ldots . - k_{n-1}\, x^{(n-1)}$$

with the k_i chosen so that the polynomial $p^n + k_{n-1} p^{n-1} + \ldots . + k_o$ has all its roots strictly in the left-half complex plane, leads to the exponentially stable dynamics

$$x^{(n)} + k_{n-1}\, x^{(n-1)} + \ldots . + k_o x = 0$$

which implies that $x(t) \rightarrow 0$. For tasks involving the tracking of a desired output $x_d(t)$, the control law

$$v = x_d^{(n)} - k_o e \; - k_2 \dot{e} - \ldots . - k_{n-1}\, e^{(n-1)} \tag{6.8}$$

(where $e(t) = x(t) - x_d(t)$ is the tracking error) leads to exponentially convergent tracking. Note that similar results would be obtained if the scalar x was replaced by a vector and the scalar b by an invertible square matrix.

One interesting application of the above control design idea is in robotics. The following example studies control design for a two-link robot. Design for more general robots is similar and will be discussed in chapter 9.

Example 6.2: Feedback linearization of a two-link robot

Figure 6.2 provides the physical model of a two-link robot, with each joint equipped with a motor for providing input torque, an encoder for measuring joint position, and a tachometer for measuring joint velocity. The objective of the control design is to make the joint positions q_1 and q_2 follow desired position histories $q_{d1}(t)$ and $q_{d2}(t)$, which are specified by the motion planning system of the robot. Such tracking control problems arise when a robot hand is required to move along a specified path, *e.g.*, to draw circles.

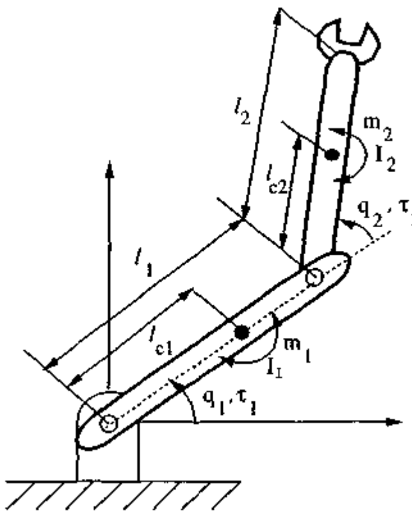


Figure 6.2 : A two-link robot

Using the well-known Lagrangian equations in classical dynamics, one can easily show that the dynamic equations of the robot is

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -h\dot{q}_2 & -h\dot{q}_1 - h\dot{q}_2 \\ h\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \tag{6.9}$$

with $\mathbf{q} = [q_1 \;\; q_2]^T$ being the two joint angles, $\boldsymbol{\tau} = [\tau_1 \;\; \tau_2]^T$ being the joint inputs, and

$$H_{11} = m_1 l_{c_1}{}^2 + I_1 + m_2 [l_1{}^2 + l_{c_2}{}^2 + 2 l_1 l_{c_2} \cos q_2] + I_2$$

$$H_{22} = m_2 l_{c_2}{}^2 + I_2$$

$$H_{12} = H_{21} = m_2 l_1 l_{c_2} \cos q_2 + m_2 l_{c_2}{}^2 + I_2$$

$$h = m_2 l_1 l_{c_2} \sin q_2$$

$$g_1 = m_1 l_{c_1} g \cos q_1 + m_2 g \,[\, l_{c_2} \cos(q_1 + q_2) + l_1 \cos q_1 \,]$$

$$g_2 = m_2 l_{c_2} g \cos(q_1 + q_2)$$

Equation (6.9) can be compactly expressed as

$$\mathbf{H}(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\,\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}$$

with **H**, **C** and **g** defined obviously. Thus, by multiplying both sides by $\mathbf{H}^{-1}$ (the invertibility of **H** is a physical property of the system, as discussed in Chapter 9), the above vector equation can be put easily in the form of (6.6), with $n = 2$, although this dynamics now involves multiple inputs and multiple outputs.

To achieve tracking control tasks, one can use the following control law

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} H_{11} \; H_{12} \\ H_{21} \; H_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} -h\dot{q}_2 & -h\dot{q}_1 - h\dot{q}_2 \\ h\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \tag{6.10}$$

where

$$\mathbf{v} = \ddot{\mathbf{q}}_d - 2\lambda\dot{\tilde{\mathbf{q}}} - \lambda^2 \tilde{\mathbf{q}}$$

with $\mathbf{v} = [\, v_1 \;\; v_2 \,]^T$ being the equivalent input, $\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{q}_d$ being the position tracking error and λ a positive number. The tracking error $\tilde{\mathbf{q}}$ then satisfies the equation

$$\ddot{\tilde{\mathbf{q}}} + 2\lambda\dot{\tilde{\mathbf{q}}} + \lambda^2 \tilde{\mathbf{q}} = \mathbf{0}$$

and therefore converges to zero exponentially. The control law (6.10) is commonly referred to as "computed torque" control in robotics. It can be applied to robots with arbitrary numbers of joints, as discussed in chapter 9. □

Note that in (6.6) we have assumed that the dynamics is linear in terms of the control input u (although nonlinear in the states). However, the approach can be easily extended to the case when u is replaced by an invertible function $g(u)$. For example, in systems involving flow control by a valve, the dynamics may be dependent on u^4 rather than directly on u, with u being the valve opening diameter. Then, by defining $w = u^4$, one can first design w similarly to the previous procedure and then compute the input u by $u = w^{1/4}$. This means that the nonlinearity is simply undone in the control computation.

When the nonlinear dynamics is not in a controllability canonical form, one may have to use algebraic *transformations* to first put the dynamics into the controllability form before using the above feedback linearization design, or to rely on partial linearization of the original dynamics, instead of full linearization. These are the topics of the next subsections. Conceptually, such transformations are not totally unfamiliar: even in the case of *linear* systems, pole placement is often most easily achieved by first putting the system in the controllability canonical form.

6.1.2 Input-State Linearization

Consider the problem of designing the control input u for a single-input nonlinear system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$$

The technique of input-state linearization solves this problem in two steps. First, one finds a state transformation $\mathbf{z} = \mathbf{z}(\mathbf{x})$ and an input transformation $u = u(\mathbf{x}, v)$ so that the nonlinear system dynamics is transformed into an equivalent *linear time-invariant* dynamics, in the familiar form $\dot{\mathbf{z}} = \mathbf{A}\,\mathbf{z} + \mathbf{b}\,v$. Second, one uses standard linear techniques (such as pole placement) to design v.

Let us illustrate the approach on a simple second-order example. Consider the system

$$\dot{x}_1 = -\,2x_1 + a\,x_2 + \sin x_1 \tag{6.11a}$$

$$\dot{x}_2 = -\,x_2 \cos x_1 + u \cos(2x_1) \tag{6.11b}$$

Even though linear control design can stabilize the system in a small region around the equilibrium point $(0, 0)$, it is not obvious at all what controller can stabilize it in a larger region. A specific difficulty is the nonlinearity in the first equation, which cannot be directly canceled by the control input u.

However, if we consider the new set of state variables

$$z_1 = x_1 \tag{6.12a}$$

$$z_2 = a x_2 + \sin x_1 \tag{6.12b}$$

then, the new state equations are

$$\dot{z}_1 = -\,2z_1 + z_2 \tag{6.13a}$$

$$\dot{z}_2 = -\,2z_1 \cos z_1 + \cos z_1 \sin z_1 + a\,u \cos(2z_1) \tag{6.13b}$$

Note that the new state equations also have an equilibrium point at $(0, 0)$. Now we see that the nonlinearities can be canceled by the control law of the form

$$u = \frac{1}{a\cos(2z_1)}(v - \cos z_1 \sin z_1 + 2z_1 \cos z_1) \tag{6.14}$$

where v is an equivalent input to be designed (equivalent in the sense that determining v amounts to determining u, and vice versa), leading to a linear input-state relation

$$\dot{z}_1 = -2z_1 + z_2 \tag{6.15a}$$

$$\dot{z}_2 = v \tag{6.15b}$$

Thus, through *the state transformation* (6.12) and input transformation (6.14), the problem of stabilizing the original nonlinear dynamics (6.11) using the original control input u has been transformed into the problem of stabilizing the new dynamics (6.15) using the new input v.

Since the new dynamics is linear and controllable, it is well known that the linear state feedback control law

$$v = -k_1 z_1 - k_2 z_2$$

can place the poles anywhere with proper choices of feedback gains. For example, we may choose

$$v = -2z_2 \tag{6.16}$$

resulting in the stable closed-loop dynamics

$$\dot{z}_1 = -2z_1 + z_2$$

$$\dot{z}_2 = -2z_2$$

whose poles are both placed at -2. In terms of the original state x_1 and x_2, this control law corresponds to the original input

$$u = \frac{1}{\cos(2x_1)}(-2ax_2 - 2\sin x_1 - \cos x_1 \sin x_1 + 2x_1 \cos x_1) \tag{6.17}$$

The original state $\mathbf{x}$ is given from $\mathbf{z}$ by

$$x_1 = z_1 \tag{6.18a}$$

$$x_2 = (z_2 - \sin z_1)/a \tag{6.18b}$$

Since both z_1 and z_2 converge to zero, the original state $\mathbf{x}$ converges to zero.

The closed-loop system under the above control law is represented in the block diagram in Figure 6.3. We can detect two loops in this control system, with the inner loop achieving the linearization of the input-state relation, and the outer loop achieving the stabilization of the closed-loop dynamics. This is consistent with (6.14), where the control input u is seen to be composed of a nonlinearity cancellation part and a linear compensation part.

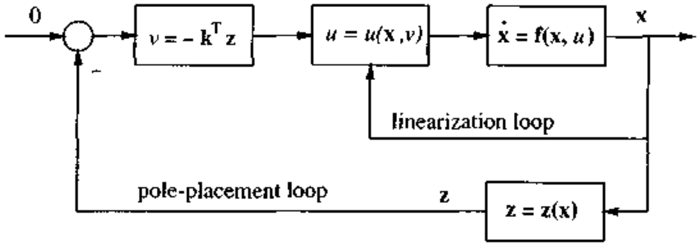


Figure 6.3 : Input-State Linearization

A number of remarks can be made about the above control law:

• The result, though valid in a large region of the state space, is not global. The control law is not well defined when $x_1 = (\pi/4 \pm k\pi/2)$, $k = 1, 2, ...$ Obviously, when the initial state is at such singularity points, the controller cannot bring the system to the equilibrium point.

• The input-state linearization is achieved by a combination of a state transformation and an input transformation, with state feedback used in both. Thus, it is a linearization by feedback, or feedback linearization. This is fundamentally different from a Jacobian linearization for small range operation, on which linear control is based.

• In order to implement the control law, the new state components (z_1, z_2) must be available. If they are not physically meaningful or cannot be measured directly, the original state **x** must be measured and used to compute them from (6.12).

• Thus, in general, we rely on the system model both for the controller design and for the computation of **z**. If there is uncertainty in the model, *e.g.*, uncertainty on the parameter a, this uncertainty will cause error in the computation of both the new state **z** and of the control input u, as seen in (6.12) and (6.14).

• Tracking control can also be considered. However, the desired motion then needs to be expressed in terms of the full new state vector. Complex computations may be needed to translate the desired motion specification (in terms of physical output variables) into specifications in terms of the new states.

With the above successful design in mind, it is interesting to extend the input-state linearization idea to general nonlinear systems. Two questions arise when one speculates such generalizations:

- What classes of nonlinear systems can be transformed into linear systems?
- How to find the proper transformations for those which can?

These questions are systematically addressed in section 6.3.

6.1.3 Input-Output Linearization

Let us now consider a tracking control problem. Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u) \tag{6.19a}$$

$$y = h(\mathbf{x}) \tag{6.19b}$$

and assume that our objective is to make the output $y(t)$ track a desired trajectory $y_d(t)$ while keeping the whole state bounded, where $y_d(t)$ and its time derivatives up to a sufficiently high order are assumed to be known and bounded. An apparent difficulty with this model is that the output y is only indirectly related to the input u, through the state variable $\mathbf{x}$ and the nonlinear state equations (6.19). Therefore, it is not easy to see how the input u can be designed to control the tracking behavior of the output y. However, inspired by the results of section 6.1.1, one might guess that the difficulty of the tracking control design can be reduced if we can *find a direct and simple relation between the system output y and the control input u.* Indeed, this idea constitutes the intuitive basis for the so-called input-output linearization approach to nonlinear control design. Let us again use an example to demonstrate this approach.

Consider the third-order system

$$\dot{x}_1 = \sin x_2 + (x_2 + 1)\, x_3 \tag{6.20a}$$

$$\dot{x}_2 = x_1{}^5 + x_3 \tag{6.20b}$$

$$\dot{x}_3 = x_1{}^2 + u \tag{6.20c}$$

$$y = x_1 \tag{6.20d}$$

To generate a direct relationship between the output y and the input u, let us differentiate the output y

$$\dot{y} = \dot{x}_1 = \sin x_2 + (x_2 + 1)\, x_3$$

Since $\dot{y}$ is still not directly related to the input u, let us differentiate again. We now obtain

$$\ddot{y} = (x_2 + 1)\ u + f_1(\mathbf{x}) \tag{6.21}$$

where $f_1(\mathbf{x})$ is a function of the state defined by

$$f_1(\mathbf{x}) = (x_1{}^5 + x_3)\,(x_3 + \cos x_2) + (x_2 + 1)\,x_1{}^2 \tag{6.22}$$

Clearly, (6.21) represents an explicit relationship between y and u. If we choose the control input to be in the form

$$u = \frac{1}{x_2 + 1}\,(v - f_1) \tag{6.23}$$

where v is a new input to be determined, the nonlinearity in (6.21) is canceled, and we obtain a simple linear double-integrator relationship between the output and the new input v,

$$\ddot{y} = v$$

The design of a tracking controller for this double-integrator relation is simple, because of the availability of linear control techniques. For instance, letting $e = y(t) - y_d(t)$ be the tracking error, and choosing the new input v as

$$v = \ddot{y}_d - k_1\, e - k_2 \dot{e} \tag{6.24}$$

with k_1 and k_2 being positive constants, the tracking error of the closed loop system is given by

$$\ddot{e} + k_2 \dot{e} + k_1 e = 0 \tag{6.25}$$

which represents an exponentially stable error dynamics. Therefore, if initially $e(0) = \dot{e}(0) = 0$, then $e(t) \equiv 0, \forall\, t \geq 0$, *i.e.*, prefect tracking is achieved; otherwise, $e(t)$ converges to zero exponentially.

Note that

- The control law is defined everywhere, except at the singularity points such that $x_2 = -1$.

• Full state measurement is necessary in implementing the control law, because the computations of both the derivative $\dot{y}$ and the input transformation (6.23) require the value of **x**.

The above control design strategy of first generating a linear input-output relation and then formulating a controller based on linear control is referred to as the *input-output linearization* approach, and it can be applied to many systems, as will be seen in section 6.4 for SISO systems and in section 6.5 for MIMO systems. If we need to differentiate the output of a system r times to generate an explicit relationship between the output y and input u, the system is said to have *relative degree* r. Thus, the system in the above example has relative degree 2. As will be shown soon, this terminology is consistent with the notion of relative degree in linear systems (excess of poles over zeros). As we shall see later, it can also be shown formally that for any controllable system of order n, it will take *at most* n differentiations of any output for the control input to appear, *i.e.*, $r \leq n$. This can be understood intuitively: if it took more than n differentiations, the system would be of order higher than n; if the control input never appeared, the system would not be controllable.

At this point, one might feel that the tracking control design problem posed at the beginning has been elegantly solved with the control law (6.23) and (6.24). However, one must remember that (6.25) only accounts for part of the closed-loop dynamics, because it has only order 2, while the whole dynamics has order 3 (the same as that of the plant, because the controller (6.23) introduces no extra dynamics). Therefore, a part of the system dynamics (described by one state component) has been rendered "unobservable" in the input-output linearization. This part of the dynamics will be called the *internal dynamics*, because it cannot be seen from the external input-output relationship (6.21). For the above example, the internal state can be chosen to be x_3 (because x_3 , y , and $\dot{y}$ constitute a new set of states), and the internal dynamics is represented by the equation

$$\dot{x}_3 = {x_1}^2 + \frac{1}{x_2 + 1}\,(\,\ddot{y}_d(t) - k_1 e - k_2 \dot{e} + f_1\,) \tag{6.26}$$

If this internal dynamics is stable (by which we actually mean that the states remain bounded during tracking, *i.e.*, stability in the BIBO sense), our tracking control design problem has indeed been solved. Otherwise, the above tracking controller is practically meaningless, because the instability of the internal dynamics would imply undesirable phenomena such as the burning-up of fuses or the violent vibration of mechanical members. Therefore, *the effectiveness of the above control design, based on the reduced-order model (6.21), hinges upon the stability of the internal dynamics.*

Let us now use some simpler examples to show that internal dynamics are stable for some systems (implying that the previous design approach is applicable), and unstable for others (implying the need for a different control design).

Example 6.3: Internal dynamics

Consider the nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^3 + u \\ u \end{bmatrix} \tag{6.27a}$$

$$y = x_1 \tag{6.27b}$$

Assume that the control objective is to make y track $y_d(t)$. Differentiation of y simply leads to the first state equation. Thus, choosing the control law

$$u = -x_2^3 - e(t) + \dot{y}_d(t) \tag{6.28}$$

yields exponential convergence of e to zero

$$\dot{e} + e = 0 \tag{6.29}$$

The same control input is also applied to the second dynamic equation, leading to the internal dynamics

$$\dot{x}_2 + x_2^3 = \dot{y}_d - e \tag{6.30}$$

which is, characteristically, non-autonomous and nonlinear. However, in view of the facts that e is guaranteed to be bounded by (6.29) and $\dot{y}_d$ is assumed to be bounded, we have

$$|\, \dot{y}_d(t) - e \,| \leq D$$

where D is a positive constant. Thus, we can conclude from (6.30) that $|x_2| \leq D^{1/3}$ (perhaps after a transient), since $\dot{x}_2 < 0$ when $x_2 > D^{1/3}$, and $\dot{x}_2 > 0$ when $x_2 < -D^{1/3}$.

Therefore, (6.28) does represent a satisfactory tracking control law for the system (6.27), given any trajectory $y_d(t)$ whose derivative $\dot{y}_d(t)$ is bounded. □

Conversely, one can easily show (Exercise 6.2) that if the second state equation in (6.27) is replaced by $\dot{x}_2 = -u$, then the resulting internal dynamics is unstable.

Finally, let us remark that, although the input-output linearization is motivated in the context of output tracking, it can also be applied to stabilization problems. For example, if $y_d(t) \equiv 0$ is the desired trajectory for the above system, the two states y and $\dot{y}$ of the closed-loop system will be driven to zero by the control law (6.28), implying the stabilization of the whole system provided that the internal dynamics is stable. In addition, two useful remarks can be made about using input-output linearization for stabilization design. First, in stabilization problems, there is no reason to restrict the choice of output $y = h(x)$ to be a physically meaningful quantity (while in tracking problems the choice of output is determined by the physical task). Any function of **x** may be used to serve as an artificial output (a designer output) to generate a linear input-output relation for the purpose of stabilization design. Second, different choices of output function leads to different internal dynamics. It is possible for one choice of output to yield a stable internal dynamics (or no internal dynamics) while another choice of output would lead to a unstable one. Therefore, one should choose, if possible, the output function to be such that the associated internal dynamics is stable.

A special case occurs when the relative degree of a system is the same as its order, *i.e.*, when the output y has to be differentiated n times (with n being the system order) to obtain a linear input-output relation. In this case, the variables $y, \dot{y}, \ldots, y^{(n-1)}$ may be used as a new set of state variables for the system, and there is no internal dynamics associated with this input-output linearization. Thus, in this case, input-output linearization leads to input-state linearization, and both state regulation and output tracking (for the particular output) can be achieved easily.

THE INTERNAL DYNAMICS OF LINEAR SYSTEMS

We must admit that it is only due to the simplicity of the system that the internal dynamics in Example 6.3 has been shown to be stable. In general, it is very difficult to directly determine the stability of the internal dynamics because it is nonlinear, non-autonomous, and coupled to the "external" closed-loop dynamics, as seen in (6.26). Although a Lyapunov or Lyapunov-like analysis may be useful for some systems, its general applicability is limited by the difficulty of finding a Lyapunov function, as discussed in chapters 3 and 4. Therefore, we naturally want to seek simpler ways of determining the stability of the internal dynamics. An examination of how the concept of internal dynamics translates in the more familiar context of *linear* systems proves helpful to this purpose.

Let us start by considering the internal dynamics of some simple linear systems.

Example 6.4: Internal dynamics in two linear systems

Consider the simple controllable and observable linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + u \\ u \end{bmatrix} \tag{6.31a}$$

$$y = x_1 \tag{6.31b}$$

where $y(t)$ is required to track a desired output $y_d(t)$. With one differentiation of the output, we simply obtain the first state equation

$$\dot{y} = x_2 + u$$

which explicitly contains u. Thus, the control law

$$u = -x_2 + \dot{y}_d - (y - y_d) \tag{6.32}$$

yields the tracking error equation

$$\dot{e} + e = 0$$

(where $e = y - y_d$) and the internal dynamics

$$\dot{x}_2 + x_2 = \dot{y}_d - e(t)$$

We see from these equations that while $y(t)$ tends to $y_d(t)$ (and $\dot{y}(t)$ tends to $\dot{y}_d(t)$), x_2 remains bounded, and so does u. Therefore, (6.32) is a satisfactory tracking controller for system (6.31).

Let us now consider a slightly different system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + u \\ -u \end{bmatrix} \tag{6.33a}$$

$$y = x_1 \tag{6.33b}$$

The same control law as above yields the same tracking error dynamics, but now leads to the internal dynamics

$$\dot{x}_2 - x_2 = e(t) - \dot{y}_d$$

This implies that x_2, and accordingly u, both go to infinity as $t \to \infty$. Therefore, (6.32) is not a suitable tracking controller for system (6.33). □

We are thus left wondering why the same tracking design method is applicable to for system (6.31) but not to system (6.33). To understand this fundamental difference between the two systems, let us consider their transfer functions, namely, for system (6.31),

$$W_1(p) = \frac{p+1}{p^2}$$

and for system (6.33),

$$W_2(p) = \frac{p-1}{p^2}$$

We see that the two systems have the same poles but different zeros. Specifically, system (6.31), for which the design has succeeded, has a *left half-plane* zero at -1, while system (6.33), for which the design has failed, has an *right half-plane* zero at 1.

The above observation (the internal dynamics is stable if the plant zeros are in the left-half plane, *i.e.*, if the plant is "minimum-phase") can actually be shown to be true for all linear systems, as we do now. This is not surprising because, for non-minimum phase systems, perfect tracking of arbitrary trajectories requires infinite control effort, as seen in Example II.2.

To keep notations simple, let us consider a third-order linear system in state-space form

$$\dot{\mathbf{z}} = \mathbf{A}\,\mathbf{z} + \mathbf{b}\,u \qquad y = \mathbf{c}^T\mathbf{z} \tag{6.34}$$

and having one zero (and hence two more poles than zeros), although the procedure can be straightforwardly extended to systems with arbitrary numbers of poles and zeros. The system's input-output linearization can be facilitated if we first transform it into the so-called companion form. To do this, we note from linear control that the input/output behavior of this system can be expressed in the form

$$y \;=\; \mathbf{c}^T(p\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{b}\,u \;=\; \frac{b_o + b_1 p}{a_o + a_1 p + a_2 p^2 + p^3}\; u \tag{6.35}$$

(where p is the Laplace variable). Thus, if we define

$$x_1 \;=\; \frac{1}{a_o + a_1 p + a_2 p^2 + p^3}\; u$$

$$x_2 \;=\; \dot{x}_1$$

$$x_3 \;=\; \dot{x}_2$$

the system can be equivalently represented in the companion form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_o & -a_1 & -a_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{6.36a}$$

$$y = [b_o \quad b_1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{6.36b}$$

Let us now perform input-output linearization based on this form. The first differentiation of the output leads to

$$\dot{y} = b_o x_2 + b_1 x_3$$

and the second differentiation leads to

$$\ddot{y} = b_o \dot{x}_2 + b_1 \dot{x}_3 = b_o x_3 + b_1(-a_o x_1 - a_1 x_2 - a_2 x_3 + u) \tag{6.37}$$

It is seen that the input u appears in the second differentiation, which means that the required number of differentiations (the relative degree) is indeed the same as the excess of poles over zeros (of course, since the input-output relation of y to u is independent of the choice of state variables, it would also take two differentiations for u to appear if we used the original state-space equations (6.34)).

Thus, the control law

$$u = (a_o x_1 + a_1 x_2 + a_2 x_3 - \frac{b_o}{b_1} x_3) + \frac{1}{b_1}(-k_1 e - k_2 \dot{e} + \ddot{y}_d) \tag{6.38}$$

where $e = y - y_d$, yields an exponentially stable tracking error

$$\ddot{e} + k_2 \dot{e} + k_1 e = 0$$

Since this is a second-order dynamics, the internal dynamics of our third-order system can be described by only one state equation. Specifically, we can use x_1 to complete the state vector, since one can easily show x_1, y, and $\dot{y}$ are related to x_1, x_2, and x_3 through a one-to-one transformation (and thus can serve as states for the system). We then easily find from (6.36a) and (6.36b) that the internal dynamics is

$$\dot{x}_1 = x_2 = \frac{1}{b_1}(y - b_o x_1)$$

that is,

$$\dot{x}_1 + \frac{b_o}{b_1} x_1 = \frac{1}{b_1} y \tag{6.39}$$

Since y is bounded ($y = e + y_d$), we see that the stability of the internal dynamics depends on the location of the zero $-b_o/b_1$ of the transfer function in (6.35). If the system is minimum phase, then the zero is in the left-half plane, which implies that the internal dynamics (6.39) is stable, independently of the initial conditions and of the magnitudes of the desired $y_d , ... , y_d^{(r)}$ (where r is the relative degree).

A classical example of the effect of a right half-plane zero is the problem of controlling the altitude of an aircraft using an elevator.

Example 6.5: Aircraft altitude dynamics

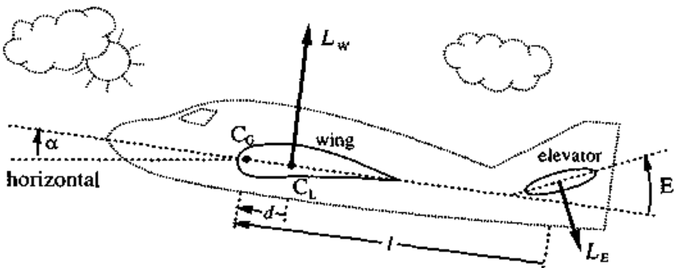


Figure 6.4 : Dynamic characteristics of an aircraft

A schematic diagram of the dynamics of an aircraft (in the longitudinal plane) is shown in Figure 6.4. The sum of the lift forces applied to the aircraft wings and body is equivalent to a single lift force L_W, applied at the "center of lift" C_L. The center of lift does not necessarily coincide with the center of mass C_G (with a positive d meaning that the center of mass is ahead of the center of lift). The mass of the aircraft is denoted by m and its moment of inertia about C_G is denoted by J. We assume that all angles are small enough to justify linear approximations, and that the forward velocity of the aircraft remains essentially constant.

The aircraft is initially cruising at a constant altitude $h = h_o$. To affect its vertical motion, the elevator (a small surface located at the aircraft tail) is rotated by an angle E. This generates a small aerodynamic force L_E on the elevator, and thus a torque about C_G. This torque creates a rotation of the aircraft about C_G, measured by an angle α. The lift force L_W applied to the wings is proportional to α , *i.e.*, $L_W = C_{ZW}\alpha$. Similarly, L_E is proportional to the angle between the horizontal and the elevator, *i.e.*, $L_E = C_{ZE}(E-\alpha)$. Furthermore, various aerodynamic forces create friction torques proportional to $\dot{\alpha}$, of the form $b\dot{\alpha}$. In summary, a simplified model of the aircraft vertical motion can be written

$$J\ddot{\alpha} + b\dot{\alpha} + (C_{ZE}\,l + C_{ZW}\,d)\alpha = C_{ZE}\,l\,E \tag{6.40a}$$

$$m\ddot{h} = (C_{ZE} + C_{ZW})\alpha - C_{ZE}\,E \tag{6.40b}$$

where the first equation represents the balance of moments and the second the balance of forces.

Remark that the open-loop stability of the first equation, which defines the dynamics of the angle α, depends on the sign of the coefficient $(C_{ZE}\, l + C_{ZW}\, d)$. In particular, the equation is open-loop stable if $d > 0$, *i.e.*, if the center of mass is ahead of the center of lift (this allows us to understand the shape of a jumbo jet, or the fact that on small aircraft passengers are first seated on the front rows).

To simplify notations, let now

$$J = 1 \quad m = 1 \quad b = 4 \quad C_{ZE} = 1 \quad C_{ZW} = 5 \quad l = 3 \quad d = 0.2.$$

The transfer functions describing the system can then be written

$$\frac{\alpha(p)}{E(p)} = \frac{3}{p^2 + 4p + 4} = \frac{3}{(p+2)^2} \tag{6.41a}$$

$$\frac{h(p)}{E(p)} = \frac{14 - 4p - p^2}{p^2(p^2 + 4p + 4)} = \frac{(6.24 + p)(2.24 - p)}{p^2(p+2)^2} \tag{6.41b}$$

where p is the Laplace variable.

At time $t = 0$, a unit step input in elevator angle E is applied. From the initial and final value theorems of linear control (or directly from the equations of motion), one can easily show that the corresponding initial and final vertical accelerations are

$$\ddot{h}(t{=}0^+) = -1 < 0 \qquad\qquad \ddot{h}(t{=}+\infty) = 3.5 > 0$$

The complete time responses in $\ddot{h}$ and h are sketched in Figure 6.5. We see that the aircraft starts in the wrong direction, then recovers. Such behavior is typical of systems with a right half-plane zero. It can be easily understood physically, and is a direct reflection of the aircraft design itself. The initial effect of the unit step in E is to create an instantaneous *downward force* on the elevator, thus creating an initial downward acceleration of the aircraft's center of mass. The unit step in elevator angle also creates a *torque* about C_G, which builds up the angle α and thus creates an increasing *upward lift force* on the wing and body. This lift force eventually takes over the downward force on the elevator. Of course, such non-minimum phase behavior is important for the pilot to know, especially when flying at low altitudes.

Let us determine the associated internal dynamics. Defining the state as $\mathbf{x} = [\,\alpha \;\; \dot{\alpha} \;\; h \;\; \dot{h}\,]^T$, the equations of motion can be written

$$\dot{x}_1 = x_2 \tag{6.42a}$$

$$\dot{x}_2 = -4x_2 - 4x_1 + 3E \tag{6.42b}$$

$$\dot{x}_3 = x_4 \tag{6.42c}$$

$$\dot{x}_4 = 6x_1 - E \tag{6.42d}$$

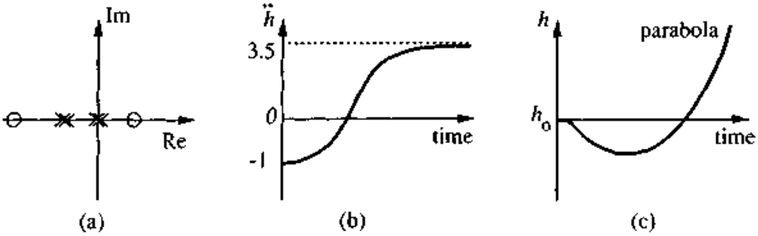


Figure 6.5 : Pole-zero plot and step responses in $\ddot{h}$ and h

The output of interest is the aircraft's altitude

$$y = x_3$$

Differentiating y until the input E appears yields

$$\ddot{y} = \ddot{x}_3 = \dot{x}_4 = 6x_1 - E$$

consistent with the fact that the transfer function (6.41b) has relative degree 2. Choose now a pole-placement control law for E

$$E = 6x_1 - \ddot{y}_d + \dot{\tilde{y}} + \tilde{y}$$

where $\tilde{y} = y - y_d$. Then

$$\ddot{\tilde{y}} + \dot{\tilde{y}} + \tilde{y} = 0 \tag{6.43}$$

The corresponding internal dynamics is

$$\dot{x}_2 = -4x_2 - 4x_1 + 3(6x_1 - \ddot{y}_d + \dot{\tilde{y}} + \tilde{y})$$

that is

$$\ddot{\alpha} + 4\dot{\alpha} - 14\alpha \;=\; 3(-\ddot{y}_d + \dot{\tilde{y}} + \tilde{y}) \tag{6.44}$$

and therefore, is unstable. Specifically, the poles on the left-hand side of (6.44) are exactly the zeros of the transfer function (6.41b). □

THE ZERO-DYNAMICS

Since for linear systems the stability of the internal dynamics is simply determined by the locations of the zeros, it is interesting to see whether this relation can be extended to nonlinear systems. To do so requires first to extend the concept of zeros to nonlinear systems, and then to determine the relation of the internal dynamics stability to this extended concept of zeros.

Extending the notion of zeros to nonlinear systems is not a trivial proposition. Transfer functions, on which linear system zeros are based, cannot be defined for nonlinear systems. Furthermore, zeros are intrinsic properties of a linear plant, while for nonlinear systems the stability of the internal dynamics may depend on the specific control input.

A way to approach these difficulties is to define a so-called *zero-dynamics* for a nonlinear system. *The zero-dynamics is defined to be the internal dynamics of the system when the system output is kept at zero by the input.* For instance, for the system (6.27), the zero-dynamics is (from (6.30))

$$\dot{x}_2 + x_2^{\,3} = 0 \tag{6.45}$$

Noticing that the specification of maintaining the system output at zero uniquely defines the required input (namely, here, u has to equal $-x_2^{\,3}$ in order to keep x_1 always equal to zero), we see that the zero-dynamics is an intrinsic property of a nonlinear system. The zero-dynamics (6.45) is easily seen to be asymptotically stable (by using the Lyapunov function $V = x_2^{\,2}$).

Similarly, for the linear system (6.34), the zero-dynamics is (from (6.39))

$$\dot{x}_1 + (b_o/b_1)\,x_1 = 0$$

Thus, in this linear system, the poles of the zero-dynamics are exactly the zeros of the system. This result is general for linear systems, and therefore, *in linear systems*, having all zeros in the left-half complex plane guarantees the global asymptotic stability of the zero-dynamics.

The reason for defining and studying the zero-dynamics is that we want to find a simpler way of determining the stability of the internal dynamics. For linear systems, the stability of the zero-dynamics implies the global stability of the internal dynamics: the left-hand side of (6.39) completely determines its stability characteristics, given that the right-hand side tends to zero or is bounded. In nonlinear systems, however, the relation is not so clear. Section 6.4 investigates this question in some detail. For stabilization problems, it can be shown that local asymptotic stability of the zero-dynamics is enough to guarantee the local asymptotic stability of the internal dynamics. Extensions can be drawn to the tracking problem. However, unlike the linear case, no results on the global stability or even large range stability can be drawn for internal dynamics of nonlinear systems, *i.e.*, only local stability is guaranteed for the internal dynamics even if the zero-dynamics is globally exponentially stable.

Similarly to the linear case, we will call a nonlinear system whose zero-dynamics is asymptotically stable an *asymptotically minimum phase* system. The concept of an exponentially minimum phase system can be defined in the same way.

Two useful remarks can be made about the zero-dynamics of nonlinear systems. First, the zero-dynamics is an intrinsic feature of a nonlinear system, which does not depend on the choice of control law or the desired trajectories. Second, examining the stability of zero-dynamics is much easier than examining the stability of internal dynamics, because the zero-dynamics only involves the internal states (while the internal dynamics is coupled to the external dynamics and desired trajectories, as seen in (6.26)).

Example 6.6: Aircraft zero-dynamics

Given (6.44), the zero-dynamics of the aircraft of Example 6.5 is

$$\ddot{\alpha} + 4\dot{\alpha} - 14\alpha = 0 \tag{6.46}$$

This dynamics is unstable, confirming that the system is non-minimum phase. The poles of the zero-dynamics are exactly the zeros of the transfer function (6.41b).

Now model (6.40) is actually the linearization of a more general *nonlinear* model, applicable at larger angles and angular rates. Since the zero-dynamics corresponding to the linearized model *simply is the linearization of the zero-dynamics corresponding to the nonlinear model*, thus the nonlinear system is also non-minimum phase, from Lyapunov's linearization method. □

To summarize, control design based on input-output linearization can be made in three steps:

- *differentiate the output y until the input u appears*
- *choose u to cancel the nonlinearities and guarantee tracking convergence*
- *study the stability of the internal dynamics*

If the relative degree associated with the input-output linearization is the same as the order of the system, the nonlinear system is fully linearized and this procedure indeed leads to a satisfactory controller (assuming that the model is accurate). If the relative degree is smaller than the system order, then the nonlinear system is only partly linearized, and whether the controller can indeed be applied depends on the stability of the internal dynamics. The study of the internal dynamics stability can be simplified locally by studying that of the zero-dynamics instead. If the zero-dynamics is unstable, different control strategies should be sought, only simplified by the fact that the transformed dynamics is partly linear.

6.2 Mathematical Tools

The objective of the rest of this chapter is to formalize and generalize the previous intuitive concepts for a broad class of nonlinear systems. To this effect, we first introduce some mathematical tools from differential geometry and topology. To limit the conceptual and notational complexity, we discuss these tools directly in the context of nonlinear dynamic systems (instead of general topological spaces). Note that this section and the remainder of this chapter represent by far the most mathematically involved part of the book. Hurried practitioners may skip it in a *first reading*, and go directly to chapters 7-9.

In describing these mathematical tools, we shall call a vector function $\mathbf{f}: \mathbf{R}^n \rightarrow \mathbf{R}^n$ a *vector field* in $\mathbf{R}^n$, to be consistent with the terminology used in differential geometry. The intuitive reason for this term is that to every vector function $\mathbf{f}$ corresponds a field of vectors in an n-dimensional space (one can think of a vector $\mathbf{f}(\mathbf{x})$ emanating from every point $\mathbf{x}$). In the following, we shall only be interested in *smooth* vector fields. By smoothness of a vector field, we mean that the function $\mathbf{f}(\mathbf{x})$ has continuous partial derivatives of any required order.

Given a smooth scalar function $h(\mathbf{x})$ of the state $\mathbf{x}$, the gradient of h is denoted by ∇h

$$\nabla h = \frac{\partial h}{\partial \mathbf{x}}$$

The gradient is represented by a *row-vector* of elements $(\nabla h)_j = \partial h / \partial x_j$. Similarly, given a vector field $\mathbf{f}(\mathbf{x})$, the Jacobian of $\mathbf{f}$ is denoted by $\nabla \mathbf{f}$

$$\nabla \mathbf{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$$

It is represented by an $n \times n$ matrix of elements $(\nabla \mathbf{f})_{ij} = \partial f_i / \partial x_j$.

LIE DERIVATIVES AND LIE BRACKETS

Given a scalar function $h(\mathbf{x})$ and a vector field $\mathbf{f}(\mathbf{x})$, we define a new scalar function $L_{\mathbf{f}}\, h$, called the Lie derivative (or simply, the derivative) of h with respect to $\mathbf{f}$.

Definition 6.1 *Let $h : \mathbf{R}^n \rightarrow \mathbf{R}$ be a smooth scalar function, and $\mathbf{f}: \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a smooth vector field on $\mathbf{R}^n$, then the Lie derivative of h with respect to $\mathbf{f}$ is a scalar function defined by* $L_{\mathbf{f}}\, h = \nabla h\ \mathbf{f}$.

Thus, the Lie derivative $L_{\mathbf{f}}h$ is simply the directional derivative of h along the direction of the vector **f**.

Repeated Lie derivatives can be defined recursively

$$L_{\mathbf{f}}^{0}\, h = h$$

$$L_{\mathbf{f}}^{i}\, h \;=\; L_{\mathbf{f}}(L_{\mathbf{f}}^{i-1}\, h) = \nabla(L_{\mathbf{f}}^{i-1}\, h)\ \mathbf{f} \qquad\qquad \text{for } i = 1, 2, \ldots.$$

Similarly, if **g** is another vector field, then the scalar function $L_{\mathbf{g}}\, L_{\mathbf{f}}\, h(\mathbf{x})$ is

$$L_{\mathbf{g}}\, L_{\mathbf{f}}\, h = \nabla(L_{\mathbf{f}}\, h)\ \mathbf{g}$$

One can easily see the relevance of Lie derivatives to dynamic systems by considering the following single-output system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

$$y = h(\mathbf{x})$$

The derivatives of the output are

$$\dot{y} \;=\; \frac{\partial h}{\partial \mathbf{x}}\,\dot{\mathbf{x}} \;=\; L_{\mathbf{f}}\, h$$

$$\ddot{y} = \frac{\partial\,[L_{\mathbf{f}}\, h]}{\partial \mathbf{x}}\,\dot{\mathbf{x}} = L_{\mathbf{f}}^{2}\, h$$

and so on. Similarly, if V is a Lyapunov function candidate for the system, its derivative $\dot{V}$ can be written as $L_{\mathbf{f}}\, V$.

Let us move on to another important mathematical operator on vector fields, the Lie bracket.

Definition 6.2 *Let* **f** *and* **g** *be two vector fields on* $\mathbf{R}^n$*. The Lie bracket of* **f** *and* **g** *is a third vector field defined by*

$$[\mathbf{f}, \mathbf{g}] \;=\; \nabla\mathbf{g}\ \mathbf{f} \;-\; \nabla\mathbf{f}\ \mathbf{g}$$

The Lie bracket $[\mathbf{f}, \mathbf{g}]$ is commonly written as $ad_{\mathbf{f}}\,\mathbf{g}$ (where *ad* stands for "adjoint"). Repeated Lie brackets can then be defined recursively by

$$ad_{\mathbf{f}}^{\,o}\,\mathbf{g} = \mathbf{g}$$

$$ad_{\mathbf{f}}^{\,i}\,\mathbf{g} = [\mathbf{f}, ad_{\mathbf{f}}^{\,i-1}\,\mathbf{g}] \qquad \text{for } i = 1,2,.....$$

Example 6.7: The system (6.11) can be written in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\,u$$

with the two vector fields **f** and **g** defined by

$$\mathbf{f} = \begin{bmatrix} -2x_1 + a x_2 + \sin x_1 \\ -\,x_2 \cos x_1 \end{bmatrix} \qquad \mathbf{g}(\mathbf{x}) = \begin{bmatrix} 0 \\ \cos(2x_1) \end{bmatrix}$$

Their Lie bracket can be computed as

$$[\mathbf{f}, \mathbf{g}] = \begin{bmatrix} 0 & 0 \\ -2\sin(2x_1) & 0 \end{bmatrix} \begin{bmatrix} -2x_1 + a x_2 + \sin x_1 \\ -x_2 \cos x_1 \end{bmatrix} - \begin{bmatrix} -2 + \cos x_1 & a \\ x_2 \sin x_1 & -\cos x_1 \end{bmatrix} \begin{bmatrix} 0 \\ \cos(2x_1) \end{bmatrix}$$

$$= \begin{bmatrix} a\cos(2x_1) \\ \cos x_1 \cos(2x_1) - 2\sin(2x_1)(-2x_1 + a x_2 + \sin x_1) \end{bmatrix} \qquad \square$$

The following lemma on Lie bracket manipulation will be useful later.

Lemma 6.1 *Lie brackets have the following properties*

(i) bilinearity:

$$[\alpha_1 \mathbf{f}_1 + \alpha_2 \mathbf{f}_2, \mathbf{g}] = \alpha_1 [\mathbf{f}_1, \mathbf{g}] + \alpha_2 [\mathbf{f}_2, \mathbf{g}]$$

$$[\mathbf{f}, \alpha_1 \mathbf{g}_1 + \alpha_2 \mathbf{g}_2] = \alpha_1 [\mathbf{f}, \mathbf{g}_1] + \alpha_2 [\mathbf{f}, \mathbf{g}_2]$$

where $\mathbf{f}, \mathbf{f}_1, \mathbf{f}_2, \mathbf{g}, \mathbf{g}_1$ *and* $\mathbf{g}_2$ *are smooth vector fields, and* α_1 *and* α_2 *are constant scalars.*

(ii) skew-commutativity:

$$[\mathbf{f}, \mathbf{g}] \; = \; -\,[\mathbf{g}, \mathbf{f}]$$

(iii) Jacobi identity:

$$L_{ad_{\mathbf{f}}\mathbf{g}}\, h \; = \; L_{\mathbf{f}} L_{\mathbf{g}}\, h - L_{\mathbf{g}} L_{\mathbf{f}}\, h$$

where $h(\mathbf{x})$ *is a smooth scalar function of* $\mathbf{x}$.

Proof: The proofs of the first two properties are straightforward (Exercise 6.6). Let us derive the third property, which can be rewritten as

$$\nabla h\,[\mathbf{f}, \mathbf{g}] = \nabla(L_{\mathbf{g}}\, h)\, \mathbf{f} - \nabla(L_{\mathbf{f}}\, h)\, \mathbf{g}$$

The left-hand side of the above equation can be expanded as

$$\nabla h\,[\mathbf{f}, \mathbf{g}] = \frac{\partial h}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{g}\right)$$

while the right-hand side can be expanded as

$$\nabla(L_{\mathbf{g}}\, h)\, \mathbf{f} - \nabla(L_{\mathbf{f}}\, h)\, \mathbf{g} = \nabla\left(\frac{\partial h}{\partial \mathbf{x}}\mathbf{g}\right)\mathbf{f} - \nabla\left(\frac{\partial h}{\partial \mathbf{x}}\mathbf{f}\right)\mathbf{g}$$

$$= \left(\frac{\partial h}{\partial \mathbf{x}}\frac{\partial \mathbf{g}}{\partial \mathbf{x}} + \mathbf{g}^T\frac{\partial^2 h}{\partial \mathbf{x}^2}\right)\mathbf{f} - \left(\frac{\partial h}{\partial \mathbf{x}}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \mathbf{f}^T\frac{\partial^2 h}{\partial \mathbf{x}^2}\right)\mathbf{g} = \frac{\partial h}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{g}\right)$$

where $\partial^2 h / \partial \mathbf{x}^2$ is the Hessian of h, a symmetric matrix. □

The *Jacobi identity* can be used recursively to obtain useful technical identities. Using it twice yields

$$\begin{aligned} L_{ad_{\mathbf{f}}^2 \mathbf{g}}\, h &= L_{ad_{\mathbf{f}}(ad_{\mathbf{f}}\mathbf{g})}\, h = L_{\mathbf{f}}\, L_{ad_{\mathbf{f}}\mathbf{g}}\, h - L_{ad_{\mathbf{f}}\mathbf{g}}\, L_{\mathbf{f}}\, h \\ &= L_{\mathbf{f}}[L_{\mathbf{f}}\, L_{\mathbf{g}}\, h - L_{\mathbf{g}}\, L_{\mathbf{f}}\, h] - [L_{\mathbf{f}}\, L_{\mathbf{g}} - L_{\mathbf{g}}\, L_{\mathbf{f}}]\, L_{\mathbf{f}}\, h \\ &= L_{\mathbf{f}}^2\, L_{\mathbf{g}}\, h - 2 L_{\mathbf{f}}\, L_{\mathbf{g}}\, L_{\mathbf{f}}\, h + L_{\mathbf{g}}\, L_{\mathbf{f}}^2\, h \end{aligned} \tag{6.47}$$

Similar identities can be obtained for higher-order Lie brackets.

DIFFEOMORPHISMS AND STATE TRANSFORMATIONS

The concept of diffeomorphism can be viewed as a generalization of the familiar concept of coordinate transformation. It is formally defined as follows:

Definition 6.3 *A function $\boldsymbol{\phi} : \mathbf{R}^n \to \mathbf{R}^n$, defined in a region Ω, is called a diffeomorphism if it is smooth, and if its inverse $\boldsymbol{\phi}^{-1}$ exists and is smooth.*

If the region Ω is the whole space $\mathbf{R}^n$, then $\boldsymbol{\phi}(\mathbf{x})$ is called a *global* diffeomorphism. Global diffeomorphisms are rare, and therefore one often looks for *local* diffeomorphisms, *i.e.*, for transformations defined only in a finite neighborhood of a given point. Given a nonlinear function $\boldsymbol{\phi}(\mathbf{x})$, it is easy to check whether it is a local diffeomorphism by using the following lemma, which is a straightforward consequence of the well-known implicit function theorem.

Lemma 6.2 *Let $\boldsymbol{\phi}(\mathbf{x})$ be a smooth function defined in a region Ω in $\mathbf{R}^n$. If the Jacobian matrix $\nabla\boldsymbol{\phi}$ is non-singular at a point $\mathbf{x} = \mathbf{x}_o$ of Ω, then $\boldsymbol{\phi}(\mathbf{x})$ defines a local diffeomorphism in a subregion of Ω.*

A diffeomorphism can be used to transform a nonlinear system into another nonlinear system in terms of a new set of states, similarly to what is commonly done in the analysis of linear systems. Consider the dynamic system described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\,u$$

$$y = h(\mathbf{x})$$

and let a new set of states be defined by

$$\mathbf{z} = \boldsymbol{\phi}(\mathbf{x})$$

Differentiation of $\mathbf{z}$ yields

$$\dot{\mathbf{z}} = \frac{\partial\boldsymbol{\phi}}{\partial\mathbf{x}}\dot{\mathbf{x}} = \frac{\partial\boldsymbol{\phi}}{\partial\mathbf{x}}(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\,u)$$

One can easily write the new state-space representation as

$$\dot{\mathbf{z}} = \mathbf{f}^*(\mathbf{z}) + \mathbf{g}^*(\mathbf{z})\,u$$

$$y = h^*(\mathbf{z})$$

where $\mathbf{x} = \boldsymbol{\phi}^{-1}(\mathbf{z})$ has been used, and the functions $\mathbf{f}^*$, $\mathbf{g}^*$ and h^* are defined obviously.

Example 6.8: A non-global diffeomorphism

The nonlinear vector function

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \boldsymbol{\phi}(\mathbf{x}) = \begin{bmatrix} 2x_1 + 5x_1x_2^{\,2} \\ 3\sin x_2 \end{bmatrix} \tag{6.48}$$

is well defined for all x_1 and x_2. Its Jacobian matrix is

$$\frac{\partial\boldsymbol{\phi}}{\partial\mathbf{x}} = \begin{bmatrix} 2 + 5x_2^{\,2} & 10\,x_1x_2 \\ 0 & 3\cos x_2 \end{bmatrix}$$

which has rank 2 at $\mathbf{x} = (0, 0)$. Therefore, Lemma 6.2 indicates that the function (6.48) defines a local diffeomorphism around the origin. In fact, the diffeomorphism is valid in the region

$$\Omega = \{(x_1, x_2)\,,\ \ |x_2| < \pi/2\}$$

because the inverse exists and is smooth for x in this region. However, outside this region, ϕ does not define a diffeomorphism, because the inverse does not uniquely exist. □

THE FROBENIUS THEOREM

The Frobenius theorem is an important tool in the formal treatment of feedback linearization for n^{th}-order nonlinear systems. It provides a necessary and sufficient condition for the solvability of a special class of partial differential equations. Before presenting the precise statement of the theorem, let us first gain a basic understanding by discussing the case $n = 3$.

Consider the set of first-order partial differential equations

$$\frac{\partial h}{\partial x_1} f_1 + \frac{\partial h}{\partial x_2} f_2 + \frac{\partial h}{\partial x_3} f_3 = 0 \tag{6.49a}$$

$$\frac{\partial h}{\partial x_1} g_1 + \frac{\partial h}{\partial x_2} g_2 + \frac{\partial h}{\partial x_3} g_3 = 0 \tag{6.49b}$$

where $f_i(x_1, x_2, x_3)$ and $g_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ are known scalar functions of x_1, x_2, x_3, and $h(x_1, x_2, x_3)$ is an unknown function. Clearly, this set of partial differential equations is uniquely defined by the two vectors $\mathbf{f} = [f_1\ \ f_2\ \ f_3]^T$, $\mathbf{g} = [g_1\ \ g_2\ \ g_3]^T$. If a solution $h(x_1, x_2, x_3)$ exists for the above partial differential equations, we shall say the set of vector fields $\{\mathbf{f}\ ,\mathbf{g}\}$ is *completely integrable*.

The question now is to determine when these equations are solvable. This is not obvious at all, *a priori*. The Frobenius theorem provides a relatively simple condition: Equation (6.49) has a solution $h(x_1, x_2, x_3)$ if, and only if, there exists scalar functions $\alpha_1(x_1, x_2, x_3)$ and $\alpha_2(x_1, x_2, x_3)$ such that

$$[\mathbf{f}, \mathbf{g}] \ = \ \alpha_1\, \mathbf{f} \ + \ \alpha_2\, \mathbf{g}$$

i.e., if the Lie bracket of **f** and **g** can be expressed as a linear combination of **f** and **g**. This condition is called the *involutivity condition* on the vector fields $\{\mathbf{f}, \mathbf{g}\}$. Geometrically it means that the vector $[\mathbf{f}, \mathbf{g}]$ is in the plane formed by the two vectors **f** and **g**. Thus, the Frobenius theorem states that the set of vector fields $\{\mathbf{f}, \mathbf{g}\}$ is completely integrable if, and only if, it is involutive. Note that the involutivity condition can be relatively easily checked, and therefore, the solvability of (6.49) can be determined accordingly.

Let us now discuss the Frobenius theorem in the general case, after giving formal definitions of complete integrability and involutivity.

Definition 6.4 *A linearly independent set of vector fields* $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ *on* $\mathbf{R}^n$ *is said to be completely integrable if, and only if, there exist* $n-m$ *scalar functions* $h_1(\mathbf{x}), h_2(\mathbf{x}), \ldots, h_{n-m}(\mathbf{x})$ *satisfying the system of partial differential equations*

$$\nabla h_i \ \mathbf{f}_j = 0 \tag{6.50}$$

where $1 \leq i \leq n-m$, $1 \leq j \leq m$, *and the gradients* ∇h_i *are linearly independent.*

Note that with the number of vectors being m and the dimension of the associated space being n, the number of unknown scalar functions h_i involved is $(n-m)$ and the number of partial differential equations is $m(n-m)$.

Definition 6.5 *A linearly independent set of vector fields* $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ *is said to be involutive if, and only if, there are scalar functions* $\alpha_{ijk} : \mathbf{R}^n \to \mathbf{R}$ *such that*

$$[\mathbf{f}_i, \mathbf{f}_j](\mathbf{x}) = \sum_{k=1}^{m} \alpha_{ijk}(\mathbf{x})\, \mathbf{f}_k(\mathbf{x}) \qquad \forall\, i, j \tag{6.51}$$

Involutivity means that if one forms the Lie bracket of any pairs of vector fields from the set $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$, then the resulting vector field can be expressed as a linear combination of the original set of vector fields. Note that

- Constant vector fields are always involutive. Indeed, the Lie bracket of two constant vectors is simply the zero vector, which can be trivially expressed as linear combination of the vector fields.

- A set composed of a single vector $\mathbf{f}$ is involutive. Indeed,

$$[\mathbf{f}, \mathbf{f}] = (\nabla \mathbf{f})\, \mathbf{f} - (\nabla \mathbf{f})\, \mathbf{f} = \mathbf{0}$$

- From Definition 6.5, checking whether a set of vector fields $\{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$ is involutive amounts to checking whether

$$\text{rank}\Big(\mathbf{f}_1(\mathbf{x}) \ \ldots\ \mathbf{f}_m(\mathbf{x})\Big) = \text{rank}\Big(\mathbf{f}_1(\mathbf{x}) \ \ldots\ \mathbf{f}_m(\mathbf{x}) \ \ [\mathbf{f}_i, \mathbf{f}_j](\mathbf{x})\Big)$$

for all $\mathbf{x}$ and all i, j.

We can now state the Frobenius theorem formally.

Theorem 6.1 (Frobenius) *Let* $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m$ *be a set of linearly independent vector fields. The set is completely integrable if, and only if, it is involutive.*

Example 6.9: Consider the set of partial differential equations

$$4x_3 \frac{\partial h}{\partial x_1} - \frac{\partial h}{\partial x_2} = 0$$

$$-x_1 \frac{\partial h}{\partial x_1} + (x_3^2 - 3x_2) \frac{\partial h}{\partial x_2} + 2x_3 \frac{\partial h}{\partial x_3} = 0$$

The associated vector fields are $\{\mathbf{f}_1, \mathbf{f}_2\}$ with

$$\mathbf{f}_1 = [4x_3 \quad -1 \quad 0]^T \qquad\qquad \mathbf{f}_2 = [-x_1 \quad (x_3^2 - 3x_2) \quad 2x_3]^T$$

In order to determine whether this set of partial differential equations is solvable (or whether $\{\mathbf{f}_1\ \mathbf{f}_2\}$ is completely integrable), let us check the involutivity of the set of vector fields $\{\mathbf{f}_1\ \mathbf{f}_2\}$. One easily finds that

$$[\mathbf{f}_1, \mathbf{f}_2] = [-12x_3 \quad 3 \quad 0]^T$$

Since $[\mathbf{f}_1, \mathbf{f}_2] = -3\mathbf{f}_1 + 0\mathbf{f}_2$, this set of vector fields is involutive. Therefore, the two partial differential equations are solvable. □

6.3 Input-State Linearization of SISO Systems

In this section, we discuss input-state linearization for single-input nonlinear systems represented by the state equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\,u \tag{6.52}$$

with $\mathbf{f}$ and $\mathbf{g}$ being smooth vector fields. We study when such systems can be linearized by state and input transformations, how to find such transformations, and how to design controllers based on such feedback linearizations.

Note that systems in the form (6.52) are said to be *linear in control* or *affine*. It is useful to point out that if a nonlinear system has the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\, w[u + \phi(\mathbf{x})]$$

with w being an *invertible* scalar function and ϕ being an arbitrary functional, a simple variable substitution $v = w[u + \phi(\mathbf{x})]$ puts the dynamics into the form of (6.52). One can design a control law for v and then compute u by inverting w, *i.e.*, $u = w^{-1}(v) - \phi(\mathbf{x})$.

DEFINITION OF INPUT-STATE LINEARIZATION

In order to proceed with a detailed study of input-state linearization, a formal definition of this concept is necessary:

Definition 6.6 *A single-input nonlinear system in the form (6.52), with* $\mathbf{f(x)}$ *and* $\mathbf{g(x)}$ *being smooth vector fields on* $\mathbf{R}^n$*, is said to be* input-state linearizable *if there exists a region* Ω *in* $\mathbf{R}^n$*, a diffeomorphism* $\phi : \Omega \rightarrow \mathbf{R}^n$*, and a nonlinear feedback control law*

$$u = \alpha(\mathbf{x}) + \beta(\mathbf{x})v \tag{6.53}$$

such that the new state variables $\mathbf{z} = \phi(\mathbf{x})$ *and the new input* v *satisfy a linear time-invariant relation*

$$\dot{\mathbf{z}} = \mathbf{A}\,\mathbf{z} + \mathbf{b}\,v \tag{6.54}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 \\ 0 & 0 & 0 & . & . & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 1 \end{bmatrix}$$

The new state $\mathbf{z}$ is called the *linearizing state*, and the control law (6.53) is called the *linearizing control law*. To simplify notations, we will often use $\mathbf{z}$ to denote not only the transformed state, but the diffeomorphism ϕ itself, *i.e.* write

$$\mathbf{z} = \mathbf{z}(\mathbf{x})$$

This slight abuse of notations should not create any confusion.

Note that the transformed linear dynamics has its $\mathbf{A}$ matrix and $\mathbf{b}$ vector of a special form, corresponding to a linear companion form. However, generality is not lost by restricting ourselves to this special linear equivalent dynamics, because any representation of a linear controllable system is equivalent to the companion form (6.54) through a linear state transformation and pole placement. Therefore, if (6.53) can be transformed into a linear system, it can be transformed into the form prescribed by (6.54) by using additional linear transformations in state and input.

We easily see from the canonical form (6.54) that feedback linearization is a special case of input-output linearization, where the output function leads to a relative degree n. This means that if a system is input-output linearizable with relative degree n, it must be input-state linearizable. On the other hand, if a system is input-state linearizable, with the first new state z_1 representing the output, the system is input-output linearizable with relative degree n. Therefore, we can summarize the relationship between input-output linearization and input-state linearization as follows:

Lemma 6.3 *An n^{th}-order nonlinear system is input-state linearizable if, and only if, there exists a scalar function $z_1(\mathbf{x})$ such that the system's input-output linearization with $z_1(\mathbf{x})$ as output function has relative degree n.*

Note, however, that the above lemma provides no guidance about how to find the desirable output function $z_1(\mathbf{x})$.

CONDITIONS FOR INPUT-STATE LINEARIZATION

At this point, a natural question is: can all nonlinear state equations in the form of (6.52) be input-state linearized? If not, when do such linearizations exist? The following theorem provides a definitive answer to that question, and constitutes one of the most fundamental results of feedback linearization theory.

Theorem 6.2 *The nonlinear system (6.52), with $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ being smooth vector fields, is input-state linearizable if, and only if, there exists a region Ω such that the following conditions hold:*

- *the vector fields $\{\mathbf{g}, ad_{\mathbf{f}}\,\mathbf{g}\,, \ldots, \; ad_{\mathbf{f}}^{\,n-1}\,\mathbf{g}\}$ are linearly independent in Ω*
- *the set $\{\mathbf{g}, ad_{\mathbf{f}}\,\mathbf{g}\,, \ldots, \; ad_{\mathbf{f}}^{\,n-2}\,\mathbf{g}\}$ is involutive in Ω*

Before proving this result, let us make a few remarks about the above conditions:

- The first condition can be interpreted as simply representing a *controllability condition* for the nonlinear system (6.52). For linear systems, the vector fields $\{\mathbf{g}, ad_{\mathbf{f}}\mathbf{g}, \ldots, ad_{\mathbf{f}}^{\,n-1}\mathbf{g}\}$ become $\{\mathbf{b}\,, \mathbf{A}\,\mathbf{b}\,, \ldots, \mathbf{A}^{n-1}\,\mathbf{b}\}$, and therefore their independence is equivalent to the invertibility of the familiar linear controllability matrix. It is also easy to show that if a system's linear approximations in a closed connected region Ω in $\mathbf{R}^n$ are all controllable, then, under some mild smoothness assumptions, the system can be driven from any point in Ω to any point in Ω. However, as mentioned in chapter 3, a nonlinear system can be controllable while its linear approximation is not. The first condition above can be shown to represent a generalized controllability condition which also accounts for such cases.

- The involutivity condition is less intuitive. It is trivially satisfied for linear systems (which have constant vector fields), but not generically satisfied in the nonlinear case.

Let us now prove the above theorem. We first state a technical lemma.

Lemma 6.4 *Let $z(\mathbf{x})$ be a smooth function in a region Ω. Then, in Ω, the set of equations*

$$L_{\mathbf{g}}\, z = L_{\mathbf{g}} L_{\mathbf{f}}\, z = \ldots = L_{\mathbf{g}} L_{\mathbf{f}}^{\,k}\, z = 0 \tag{6.55a}$$

is equivalent to

$$L_{\mathbf{g}}\, z = L_{ad_{\mathbf{f}}\mathbf{g}}\, z = \ldots = L_{ad_{\mathbf{f}}^{\,k}\mathbf{g}}\, z = 0 \tag{6.55b}$$

for any positive integer k.

Proof: Let us show that (6.55a) implies (6.55b).

When $k = 0$, the result is obvious. When $k = 1$, we have from Jacobi's identity (Lemma 6.1)

$$L_{ad_{\mathbf{f}}\mathbf{g}}\, z \;=\; L_{\mathbf{f}} L_{\mathbf{g}}\, z - L_{\mathbf{g}} L_{\mathbf{f}}\, z \;=\; 0 - 0 \;=\; 0$$

When $k = 2$, we further have, using Jacobi's identity twice as in (6.47)

$$L_{ad_{\mathbf{f}}^{\,2}\mathbf{g}}\, z \;=\; L_{\mathbf{f}}^{2} L_{\mathbf{g}}\, z - 2 L_{\mathbf{f}} L_{\mathbf{g}} L_{\mathbf{f}}\, z + L_{\mathbf{g}} L_{\mathbf{f}}^{2}\, z \;=\; 0 - 0 + 0 \;=\; 0$$

Repeating this procedure, we can show by induction that (6.55a) implies (6.55b) for any k.

One proceeds similarly to show that (6.55b) implies (6.55a) (by using Jacobi's identity the other way around). □

We are now ready for the proof of Theorem 6.2 itself.

Proof of Theorem 6.2: Let us first prove the *necessity* of the conditions. Assume that there exist a state transformation $\mathbf{z} = \mathbf{z}(\mathbf{x})$ and an input transformation $u = \alpha(\mathbf{x}) + \beta(\mathbf{x})\, v$ such that $\mathbf{z}$ and v satisfy (6.54). Expanding the first line of (6.54), we obtain

$$\dot{z}_1 = \frac{\partial z_1}{\partial \mathbf{x}}\,(\mathbf{f} + \mathbf{g}\, u) = z_2$$

Proceeding similarly with the other components of $\mathbf{z}$ leads to a set of partial differential equations

$$\frac{\partial z_1}{\partial \mathbf{x}}\,\mathbf{f} \,+\, \frac{\partial z_1}{\partial \mathbf{x}}\;\mathbf{g}\, u \;=\; z_2$$

$$\frac{\partial z_2}{\partial \mathbf{x}}\,\mathbf{f} \,+\, \frac{\partial z_2}{\partial \mathbf{x}}\;\mathbf{g}\, u \;=\; z_3$$

$$\ldots$$

$$\frac{\partial z_n}{\partial \mathbf{x}}\,\mathbf{f} \,+\, \frac{\partial z_n}{\partial \mathbf{x}}\;\mathbf{g}\, u \;=\; v$$

Since $z_1, \ldots, z_n$ are independent of u, while v is not, we conclude from the above equations that

$$L_{\mathbf{g}} z_1 = L_{\mathbf{g}} z_2 = \dots = L_{\mathbf{g}} z_{n-1} = 0 \qquad\qquad L_{\mathbf{g}} z_n \neq 0 \tag{6.56a}$$

$$L_{\mathbf{f}} z_i = z_{i+1} \qquad i = 1, 2, \dots, n-1 \tag{6.56b}$$

The above equations on the z_i can be compressed into a set of constraint equations on z_1 alone. Indeed, using Lemma 6.4, equation (6.56a) implies that

$$\nabla z_1 \, ad_{\mathbf{f}}^{\,k} \, \mathbf{g} \;=\; 0 \qquad k = 0, 1, 2, \dots, n-2 \tag{6.57a}$$

Furthermore by proceeding as in the proof of Lemma 6.4, we can show

$$\nabla z_1 \, ad_{\mathbf{f}}^{\,n-1} \, \mathbf{g} \;=\; (-1)^{n-1} \, L_{\mathbf{g}} \, z_n$$

This implies that

$$\nabla z_1 \, ad_{\mathbf{f}}^{\,n-1} \, \mathbf{g} \;\neq\; 0 \tag{6.57b}$$

The first property we can now infer from (6.57) is that the vector fields $\mathbf{g}, ad_{\mathbf{f}}\,\mathbf{g}, \dots, ad_{\mathbf{f}}^{\,n-1}\,\mathbf{g}$ must be linearly independent. Indeed, if for some number i ($i \leq n-1$), there existed scalar functions $\alpha_1(\mathbf{x}), \dots, \alpha_{i-1}(\mathbf{x})$ such that

$$ad_{\mathbf{f}}^{\,i}\,\mathbf{g} = \sum_{k=0}^{i-1} \alpha_k \, ad_{\mathbf{f}}^{\,k}\,\mathbf{g}$$

we would have

$$ad_{\mathbf{f}}^{\,n-1}\,\mathbf{g} \;=\; \sum_{k=n-i-1}^{n-2} \alpha_k \, ad_{\mathbf{f}}^{\,k}\,\mathbf{g}$$

This, together with (6.57a), would imply that

$$\nabla z_1 \;\, ad_{\mathbf{f}}^{\,n-1}\,\mathbf{g} \;=\; \sum_{k=n-i-1}^{n-2} \alpha_k \, \nabla z_1 \, ad_{\mathbf{f}}^{\,k}\,\mathbf{g} \;=\; 0$$

a contradiction to (6.57b).

The second property we can infer from (6.57) is that the vector fields are involutive. This follows from the existence of a scalar function z_1 satisfying the $n-1$ partial differential equations in (6.57a), and from the necessity part of the Frobenius theorem. Thus, we have completed the necessity part of the proof of Theorem 6.2.

Let us now prove that the two conditions in Theorem 6.2 are also *sufficient* for the input-state linearizability of the nonlinear system in (6.52), *i.e.*, that we can find a state transformation and an input transformation such that (6.54) is satisfied. The reasoning is as follows. Since the involutivity condition is satisfied, from Frobenius theorem there exists a non-zero scalar function $z_1(\mathbf{x})$ satisfying

$$L_{\mathbf{g}}\, z_1 = L_{ad_{\mathbf{f}}\mathbf{g}}\, z_1 = \ldots = L_{ad_{\mathbf{f}}^{n-2}\mathbf{g}}\, z_1 = 0 \tag{6.58}$$

From Lemma 6.4, the above equations can be written

$$L_{\mathbf{g}}\, z_1 = L_{\mathbf{g}} L_{\mathbf{f}}\, z_1 = \ldots = L_{\mathbf{g}} L_{\mathbf{f}}^{n-2}\, z_1 = 0 \tag{6.59}$$

This means that if we use $\mathbf{z} = [\, z_1 \;\; L_{\mathbf{f}} z_1 \;\; \ldots \;\; L_{\mathbf{f}}^{\,n-1} z_1 \,]^T$ as a new set of state variables, the first $n-1$ state equations verify

$$\dot{z}_k \;=\; z_{k+1} \qquad k = 1, \ldots, n-1$$

while the last state equation is

$$\dot{z}_n \;=\; L_{\mathbf{f}}^{\,n} z_1 + L_{\mathbf{g}} L_{\mathbf{f}}^{\,n-1} z_1\, u \tag{6.60}$$

Now the question is whether $L_{\mathbf{g}} L_{\mathbf{f}}^{\,n-1} z_1$ can be equal to zero. Since the vector fields $\{\, \mathbf{g}, ad_{\mathbf{f}}\mathbf{g}, \ldots, ad_{\mathbf{f}}^{\,n-1}\mathbf{g} \,\}$ are linearly independent in Ω, and noticing, as in the proof of Lemma 6.4, that (6.58) also leads to

$$L_{\mathbf{g}} L_{\mathbf{f}}^{\,n-1} z_1 \;=\; (-1)^{n-1}\, L_{ad_{\mathbf{f}}^{n-1}\mathbf{g}}\, z_1$$

we must have

$$L_{ad_{\mathbf{f}}^{n-1}\mathbf{g}}\, z_1(\mathbf{x}) \;\neq\; 0 \qquad\qquad \forall\, \mathbf{x} \in \Omega \tag{6.61}$$

Otherwise, the non-zero vector ∇z_1 would satisfy

$$\nabla z_1 \;\; [\, \mathbf{g} \;\;\; ad_{\mathbf{f}}\mathbf{g} \;\;\; \ldots \;\;\; ad_{\mathbf{f}}^{\,n-1}\mathbf{g} \,] \;=\; 0$$

and thus would be orthogonal to n linearly independent vectors, a contradiction.

Therefore, by taking the control law to be

$$u \;=\; (\, - L_{\mathbf{f}}^{\,n} z_1 + v)/(L_{\mathbf{g}} L_{\mathbf{f}}^{\,n-1} z_1)$$

equation (6.60) simply becomes

$$\dot{z}_n = v$$

which shows that the input-output linearization of the nonlinear system has been achieved. □

HOW TO PERFORM INPUT-STATE LINEARIZATION

Based on the previous discussion, the input-state linearization of a nonlinear system can be performed through the following steps:

- Construct the vector fields $\mathbf{g}, ad_{\mathbf{f}}\,\mathbf{g}, \ldots, ad_{\mathbf{f}}^{\,n-1}\mathbf{g}$ for the given system
- Check whether the controllability and involutivity conditions are satisfied

• If both are satisfied, find the first state z_1 (the output function leading to input-output linearization of relative degree n) from equations (6.58), *i.e.*,

$$\nabla z_1\, ad_{\mathbf{f}}^{\,i}\mathbf{g} = 0 \qquad i = 0, \ldots, n-2 \tag{6.62a}$$

$$\nabla z_1\, ad_{\mathbf{f}}^{\,n-1}\mathbf{g} \neq 0 \tag{6.62b}$$

• Compute the state transformation $\mathbf{z}(\mathbf{x}) = [\, z_1 \;\; L_{\mathbf{f}}\, z_1 \;\; \ldots \;\; L_{\mathbf{f}}^{\,n-1} z_1 \,]^T$ and the input transformation (6.53), with

$$\alpha(\mathbf{x}) = -\frac{L_{\mathbf{f}}^{\,n} z_1}{L_{\mathbf{g}}\, L_{\mathbf{f}}^{\,n-1} z_1} \tag{6.63a}$$

$$\beta(\mathbf{x}) = \frac{1}{L_{\mathbf{g}}\, L_{\mathbf{f}}^{\,n-1} z_1} \tag{6.63b}$$

Let us now demonstrate the above procedure on a simple physical example [Marino and Spong, 1986; Spong and Vidyasagar, 1989].

Example 6.10: Consider the control of the mechanism in Figure 6.6, which represents a link driven by a motor through a torsional spring (a single-link flexible-joint robot), in the vertical plane. Its equations of motion can be easily derived as

$$I\ddot{q}_1 + MgL\sin q_1 + k(q_1 - q_2) = 0 \tag{6.64a}$$

$$J\ddot{q}_2 - k(q_1 - q_2) = u \tag{6.64b}$$

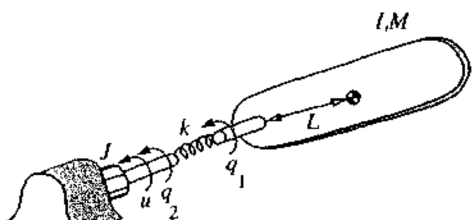


Figure 6.6 : A flexible-joint mechanism

Because nonlinearities (due to gravitational torques) appear in the first equation, while the control input u enters only in the second equation, there is no obvious way of designing a large range controller. Let us now consider whether input-state linearization is possible.

First, let us put the system's dynamics in a state-space representation. Choosing the state vector as

$$\mathbf{x} = [q_1 \;\; \dot{q}_1 \;\; q_2 \;\; \dot{q}_2]^T$$

the corresponding vector fields $\mathbf{f}$ and $\mathbf{g}$ can be written

$$\mathbf{f} = [\, x_2 \quad -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \quad x_4 \quad \frac{k}{J}(x_1 - x_3)\,]^T$$

$$\mathbf{g} = [\,0 \;\; 0 \;\; 0 \;\; \frac{1}{J}\,]^T$$

Second, let us check the controllability and involutivity conditions. The controllability matrix is obtained by simple computation

$$[\,\mathbf{g} \quad ad_{\mathbf{f}}\mathbf{g} \quad ad^2{}_{\mathbf{f}}\mathbf{g} \quad ad^3{}_{\mathbf{f}}\mathbf{g}\,] = \begin{bmatrix} 0 & 0 & 0 & -\frac{k}{IJ} \\ 0 & 0 & \frac{k}{IJ} & 0 \\ 0 & -\frac{1}{J} & 0 & \frac{k}{J^2} \\ \frac{1}{J} & 0 & -\frac{k}{J^2} & 0 \end{bmatrix}$$

It has rank 4 for $k > 0, IJ < \infty$. Furthermore, since the vector fields $\{\mathbf{g}, ad_{\mathbf{f}}\mathbf{g}, ad_{\mathbf{f}}{}^2\mathbf{g}\}$ are constant, they form an involutive set. Therefore, the system in (6.64) is input-state linearizable.

Third, let us find out the state transformation $\mathbf{z} = \mathbf{z}(\mathbf{x})$ and the input transformation $u = \alpha(\mathbf{x}) + \beta(\mathbf{x})\,v$ so that input-state linearization is achieved. From (6.62), and given the above expression of the controllability matrix, the first component z_1 of the new state vector $\mathbf{z}$ should satisfy

$$\frac{\partial z_1}{\partial x_2} = 0 \qquad \frac{\partial z_1}{\partial x_3} = 0 \qquad \frac{\partial z_1}{\partial x_4} = 0 \qquad \frac{\partial z_1}{\partial x_1} \neq 0$$

Thus, z_1 must be a function of x_1 only. The simplest solution to the above equations is

$$z_1 = x_1 \tag{6.65a}$$

The other states can be obtained from z_1

$$z_2 = \nabla z_1 \, \mathbf{f} = x_2 \tag{6.65b}$$

$$z_3 = \nabla z_2 \, \mathbf{f} = -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \tag{6.65c}$$

$$z_4 = \nabla z_3\, \mathbf{f} = -\frac{MgL}{I} x_2 \cos x_1 - \frac{k}{I}(x_2 - x_4) \tag{6.65d}$$

Accordingly, the input transformation is

$$u = (v - \nabla z_4\, \mathbf{f}) / (\nabla z_4\, \mathbf{g})$$

which can be written explicitly as

$$u = \frac{IJ}{k}(v - a(\mathbf{x})) \tag{6.66}$$

where

$$a(\mathbf{x}) = \frac{MgL}{I} \sin x_1 \,(x_2^2 + \frac{MgL}{I} \cos x_1 + \frac{k}{I}) + \frac{k}{I}(x_1 - x_3)\,(\frac{k}{I} + \frac{k}{J} + \frac{MgL}{I} \cos x_1)$$

As a result of the above state and input transformations, we end up with the following set of linear equations

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = z_3$$
$$\dot{z}_3 = z_4$$
$$\dot{z}_4 = v$$

thus completing the input-state linearization.

Finally, note that

- The above input-state linearization is actually global, because the diffeomorphism $\mathbf{z}(\mathbf{x})$ and the input transformation are well defined everywhere. Specifically, the inverse of the state transformation (6.65) is

$$x_1 = z_1$$
$$x_2 = z_2$$
$$x_3 = z_1 + \frac{I}{k}\left(z_3 + \frac{MgL}{I} \sin z_1\right)$$
$$x_4 = z_2 + \frac{I}{k}\left(z_4 + \frac{MgL}{I}\, z_2 \cos z_1\right)$$

which is well defined and differentiable everywhere. The input transformation (6.66) is also well defined everywhere, of course.

- In this particular example, the transformed variables have physical meanings. We see that

z_1 is the link position, z_2 the link velocity, z_3 the link acceleration, and z_4 the link jerk. This further illustrates our earlier remark that the complexity of a nonlinear physical model is strongly dependent on the choice of state variables.

• In hindsight, of course, we also see that the same result could have been derived simply by differentiating equation (6.64a) twice, *i.e.*, from the input-output linearization perspective of Lemma 6.3. □

Note that inequality (6.62b) can be replaced by the normalization equation

$$\nabla z_1 \, ad_{\mathbf{f}}^{\,n-1}\mathbf{g} = 1$$

without affecting the input-state linearization. This equation and (6.62a) constitute a total of n linear equations,

$$[\, ad_{\mathbf{f}}^{0}\mathbf{g} \quad ad_{\mathbf{f}}^{1}\mathbf{g} \quad \dots \quad ad_{\mathbf{f}}^{n-2}\mathbf{g} \quad ad_{\mathbf{f}}^{n-1}\mathbf{g}\,] \begin{bmatrix} \partial z_1/\partial x_1 \\ \partial z_1/\partial x_2 \\ \dots\dots \\ \partial z_1/\partial x_{n-1} \\ \partial z_1/\partial x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}$$

Given the independence condition on the vector fields, the partial derivatives $\partial z_1/\partial x_1$, ... , $\partial z_1/\partial x_n$ can be computed uniquely from the above equations. The state variable z_1 can then be found, in principle, by sequentially integrating these partial derivatives. Note that analytically solving this set of partial differential equations for z_1 may be a nontrivial step (although numerical solutions may be relatively easy due to the recursive nature of the equations).

CONTROLLER DESIGN BASED ON INPUT-STATE LINEARIZATION

With the state equation transformed into a linear form, one can easily design controllers for either stabilization or tracking purposes. A stabilization example has already been provided in the intuitive section 6.1, where v is designed to place the poles of the equivalent linear dynamics, and the physical input u is then computed using the corresponding input transformation. One can also design tracking controllers based on the equivalent linear system, provided that the desired trajectory can be expressed in terms of the first linearizing state component z_1.

Consider again the flexible link example. Its equivalent linear dynamics can be expressed as

$$z_1^{(4)} = v$$